FUNDAMENTALS OF HIGH SCHOOL MATHEMATICS

RUGG-CLARK

FUNDAMENTALS OF HIGH SCHOOL MATHEMATICS

A TEXTBOOK DESIGNED TO FOLLOW ARITHMETIC

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The purpose of this house is to publish books that apply the world's knowledge to the world's needs. Public-school texts in mathematics should be designed in terms of careful studies of social needs as well as in terms of methods by which pupils learn. This book on general mathematics is the first of the Rugg-Clark Mathematics Texts, and it will be followed by texts for Junior High Schools and for Elementary Schools. Each of these texts will embody the result of years of scientific investigation and experimental teaching

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SCIENTIFIC METHOD IN THE CONSTRUCTION OF SCHOOL TEXTBOOKS

The traditional course of study in high school mathematics needs to be completely reconstructed. Five years of intensive investigation * have established striking inadequacies in the first-year course: both what we teach and how we teach it. *First:* more than half of the conventional first-year course will never be used by the vast majority of our pupils. *Second:* the claims which we make that we are training pupils "to think intelligently" will be difficult, if not impossible, to substantiate. The very content and organization of the course tends to inhibit this. Most of it provides little or no opportunity for training in "problem-solving." Our courses of study emphasize habit-formation and rote memory, and these courses are almost exactly determined by the textbook. *Finally:* standardized tests given to more than 100 high schools show clearly that neither in securing formal skill nor in developing powers of analytical thinking is our instruction satisfactory.

Striking needs: (1) courses of study based on a clear-cut program; (2) a real psychology of how children learn mathematics, expressed through new types of "wordy" textbooks and manuals of method. Courses must be constructed on principles of social worth and "thinking" outcomes that have been definitely established. Current courses have not been so organized. The slate should be wiped clean and a new course organized on a clear-cut program. No material should be included which cannot be defended either on the basis of social worth, or upon the probability of relatively worth-while thought power. Furthermore, the present course, as it came down from its former position in the upper college curriculum, has retained that emphasis upon rigorous, logical organization and upon brief alge-

* The evidence which was collected is reported in detail in *Scientific Method in the Reconstruction of Ninth Grade Mathematics*, by H. O. Rugg and J. R. Clark; Department of Education, Bureau of Publications, University of Chicago, 1918.

braic symbolism which characterizes the thinking of "mathematicians." Children, however, do their thinking in terms of detailed word-symbols; only laboriously do they take on the more abbreviated methods of symbolic thinking which are typical of mathematical manipulation. The solution is clear: we need "wordy" textbooks — textbooks in which the transition from thinking in detailed word-symbols to that in abbreviated letter or algebraic symbols and in their manipulation is made so gradually as to keep, step by step, just ahead of the pupil's mental advance. "Gradation" of subject matter must receive a real psychological significance in the mind of the textbook writer and of the teacher. Basic to all these statements, however, is the need for the development of a real psychological analysis of how children learn mathematics.

These facts, coupled with the accumulation of dissatisfaction and denunciation on the part of both school administrators and lay critics of current mathematical courses, point to the pressing need of setting in motion a program for completely reconstructing our whole scheme of mathematical instruction.

"FUNDAMENTALS OF HIGH SCHOOL MATHEMATICS" IS A TRANSITION BOOK, INTENDED TO FILL AN IMMEDIATE NEED FOR REQUIRED MATHEMATICS IN THE NINTH GRADE

The **pressing need** is for a complete rebuilding of the material of the ninth school year. There is much evidence that the "required" mathematics of the future high school course will stop with the ninth year. Hence, the course of study submitted herewith is based on the assumption that the mathematics of the ninth grade will be the last year required of all children. Under the present organization of the seventh- and eighth-grade courses, therefore, this new ninth-grade course must include all the fundamental mathematical notions and devices which can be taught in one school year and to children of that grade of

maturity. Furthermore, it is based on the assumption — yes, the demand — that *there be one year of mathematics beyond arithmetic required of all children* who remain to our ninth grade.

Although the immediate need is an acceptable "transition" course for the ninth grade, *the permanent need is for a scientifically constructed unit course of study for the junior high school grades, seventh, eighth, and ninth.* The authors are conducting investigations which are basic to the development of such a course of study. The course will be published only after thorough scientific study and experimentation in the classroom. The present book, *Fundamentals of High School Mathematics*, therefore, they regard as a "transition" book to fill a pressing need in "first-year mathematics," much of the material of which will shortly be found distributed with other materials over the eighth and ninth school grades.

"FUNDAMENTALS OF HIGH SCHOOL MATHEMATICS" IS CONSTRUCTED ON A DEFINITE SET OF PRINCIPLES

I. THE SELECTION OF SUBJECT MATTER ON THE CRITERIA OF SOCIAL WORTH AND "THINKING" OUTCOMES

It is the judgment of the present writers that a course of study should rest upon a definitely founded program of *selection* and *arrangement* of subject matter. Courses of study should, periodically at least, be recast, not by elimination from or accretion to the stock course, but rather by spreading out all the curriculum materials (mathematical in this case) in full view, and taking them into the course in the degree to which they satisfy the criteria which have been set up. Doing this in mathematics, one includes, *first*, selected material which is now in the traditional first-year course, and *second*, adds much material which is either not taught at all to high school students or is taught to only a limited portion in some advanced year.

Two principles should control the selection of subject matter: *first, social worth* — material to be included must prove of social worth (in school, in the home, in the occupation, or in leisure activities) to a definite portion of our student body; *second,* "*thinking value*" — it must be possible to show for material whose social utility is doubtful that a thorough grasp of those quantitative principles which are needed for sound and complete thinking in life will be hampered by their exclusion.

Acting on these two criteria, the authors' investigations of "how much mathematics" should be experimented with in the classroom and finally included in the course, have demanded the exclusion of at least 35 per cent of the material of current first-year algebra. The time usually given to formal skill in handling polynomials (with the four fundamental operations), highest common factor, least common multiple, the mastery of 7 to 17 cases in "special products" and factoring, skill in manipulating complicated forms of fractions and fractional equations, can have no justification in terms of *this criterion of social worth.*

Neither does it find a place because of its "thinking" value. This is all "formal skill" material, provides no opportunity for training in the technique of reasoning, and certainly does not contribute to a grasp of "scientific law."

1. **What to include in the new course on the criterion of social worth.** The course must include, even on the basis of social worth alone, training in (*a*) the use of letters to represent numbers; (*b*) the use of the simple equation; (*c*) the construction and evaluation of formulas; (*d*) the finding of unknown distances by means of (1) scale drawings, (2) the principle of similarity in triangles, (3) the use of the properties of the right triangle (ratios of the sides as expressed in the Hypotenuse Rule, and in the cosine and the tangent of an angle); (*e*) the preparation and use of statistical tables and graphs to represent and compare quantities (this includes the grasp of elementary statistical measures).

2. **The application of the "thinking" criterion.** So much from the social criterion. The writers are of that group, however, that regard "*thinking*" *outcomes* as *coördinate in importance with the more commonly accepted* "*social utility.*"

The primary function of mathematical instruction. Their thesis in constructing this course of study is this: **The central element in human thinking is the ability to see relationships clearly. In the same way the primary function of a high school course in mathematics is to give ability to recognize relationships between magnitudes, to represent such relationships economically by means of symbols, and to determine such relationships.** To carry out this aim the course of study, therefore, should be organized in such a way as *to develop ability in the intelligent use of the equation, the formula, methods of graphic representation, and the properties of the more important space forms in* **the expression and determination of relationships.**

Note the importance of that last phrase — "*in the expression and determination of relationships*"; *i.e.* of "law." The course of study in mathematics must not only contain socially worthwhile material but it must coöperate with other subjects in the curriculum in providing training in the development of the scientific attitude. This, in turn, can come only through constant practice in meeting real problem-situations, and in a grasp of the principle of "functionality," *i.e.* of "dependence" or, more concretely, of RELATIONSHIP.

But practice in meeting problem-situations demands that the whole course be constructed around a core of problem-solving. Unfortunately, algebra courses have been deprived of most of their "training" value. An emphasis upon formalism, drill, the routine practice in manipulation of meaningless symbols, and lack of genuine motive are typical examples of the way in which we have hampered teachers in the development of problem-solving abilities. The general practice of devoting 80 per cent of the exercise-material to these formal drill examples, leaving

only 20 per cent for the verbal problem, — which of all types provides most completely opportunity for "thinking," — has been radically modified in the course submitted in this book. **The entire course has been organized around a central core of "problem-solving."** Even the purely formal materials themselves have been so organized, wherever possible, as to provide an opportunity for real thinking and not mere habit formation.

Teaching children to understand and express "law." Our courses of study have failed, generally, to the present time, to give our high school students a grasp of functionality, *i.e.* of scientific "law," and how to express it. Thus, both the basic mathematical purpose of the course and the foundational "thinking" purpose of the course have not been fulfilled. Hence the importance, in this complete recasting of the course, of attempting to build it in such a way as to contribute constantly to ability in the expression and determination of RELATIONSHIP. Chapters VIII and XVI and scattered problem-material throughout every chapter provide the type of definite training that the writers' experimentation has shown is necessary and can be given to help bring about the desired outcome in a one-year course.

II. THE ARRANGEMENT OF THE SUBJECT MATTER IN A TEXTBOOK

The psychological criterion and the principle of mathematical sequence control the arrangement. Two principles only should control the sequence and gradation of subject matter: (1) mathematical sequence; (2) learning difficulty. The two principles operate to control the organization of the material in this book. Two striking illustrations can be given of the application of the psychological criterion. The first has to do with the teaching of special products and factoring; the second, that of signed numbers. A further and basic example can be found in the new method of writing a mathematics book — *i.e.* the "wordy" textbook.

The writers have experimented in the classroom with this reconstructed course of study. Each has taught first-year classes under the critical observation and comment of the other. The work has eventuated in an important body of material concerning "how children learn mathematics." (This material will be presented in a book, *The Psychology and Teaching of Junior High School Mathematics*, some time during the school year 1919–1920.) It has been clearly demonstrated that special products and factoring can be satisfactorily taught and habituated in 7 class periods, — these contrasted with the 30 class periods of the common practice of the day.

Similarly our experimental teaching of the course with signed numbers introduced both early and late has supplied what appears to the writers, and to more than 50 coöperating high schools, to be conclusive evidence for using negative numbers only in the second half of the course. This necessitates the complete reconstruction of the order and complexity of the material of the first half year, and leads to a type of gradation that satisfies the basic psychological criterion for arranging the subject matter of courses; namely, the mental content of the course must keep just one step in advance of the developing content of the pupil's mind.

A SUMMARY OF DISTINCTIVE CONTRIBUTIONS TO THE TEACHING OF MATHEMATICS ILLUSTRATED BY THE CONTENT AND ARRANGEMENT OF "FUNDAMENTALS OF HIGH SCHOOL MATHEMATICS"

A scientific program has been developed for constructing courses of study, the steps of which are necessary for the sound design of textbooks that will be relatively permanent in our generation. It has two striking characteristics:

(*A*) **The content of the course selected** so as to satisfy rigorous criteria of either social worth or definitely established "thinking" value, or both. This procedure is in contrast to

that of elimination from, addition to, or rearrangement of a stock course of study. In this book it has led to (1) vast economy of time by excluding non-essential operations and forms; (2) introduction of new material not commonly attempted or successfully taught in the first-year course: *e.g.* (*a*) the use of statistical measures, tables, and graphs to represent and compare quantities; (*b*) **the organization of a whole course about the central theme of relationship ("functionality") and the systematic organization of three methods of representing and determining relationship** — the graphic method, the tabular method, and the equational or formula method; (*c*) systematic teaching of methods of indirect measurement — *i.e.* the finding of unknown distances by scale drawings, similar triangles, and the use of the properties of the right triangle.

(*B*) **Psychological arrangement of textbooks secured only by real classroom experimentation and coöperative teaching by many teachers.** For the present very general practice of making school textbooks at the desk and without careful classroom studies of how children learn, the authors substituted a unique experimental procedure. Following a year of detailed experimentation in their own classes the tentative course of study was printed, sold at cost by the authors, and taught by teachers of very typical experience and training in 62 high schools. From these teachers a most searching criticism of the organization of the book was obtained. In the light of this detailed analysis the book has been completely rebuilt. The writers submit it now for general use in the belief that it is constructed in accordance with the ways in which children learn and that it fits the necessary administrative demands of the classroom. Several illustrations can be given of the contributions that can be made by this type of experimentation:

(1) A new type of textbook — a "wordy" textbook — one that will teach itself. There is much evidence for the conclusion that the exposition of the text develops so gradually in accord-

ance with the way in which children "learn" that the rank and file of pupils can read it and work the problems unaided. This provides not only a helpful and complete guide for the inexperienced teacher (which is most pertinently needed in all the school subjects) but leaves the classroom time of the experienced teacher relatively free for the introduction of important supplementary material.

(2) The postponement of signed numbers to the second half of the course, providing for a very gradual and unique development of the subject matter of the first half.

(3) The development of a new method of teaching special products and factoring which saves at least 20 school days in the course.

(4) Graphic representation is an integral part of the course, and is treated throughout as a method of representing quantity — not as an isolated operation.

HOW THE COURSE PREPARES FOR THE STUDY OF ADVANCED MATHEMATICS

A sound course of study in ninth-grade mathematics must prepare adequately for conventional third-semester algebra and plane and solid geometry and trigonometry which will be taken by an increasing number of students. The striking fact is that the construction of a text on a scientific program results in a course that prepares for such courses even more completely than the conventional "first-year algebra."

The formal material that has been excluded not only has no "social" utility outside the school, but it is not used in these advanced courses. The new material that has been added is direct preparation for them. For example, the three chapters that discuss the finding of unknown distances, the chapters on expressing and determining relationship, and direct and inverse variation, develop intensively concepts and tools which are in-

dispensable to the mastery of geometry, trigonometry, and the higher courses. Furthermore, such a procedure gives a far more thorough training in the basic algebraic skills. Thus the course satisfies the necessary administrative requirement that it must fit into an established sequence of school courses.

THE RELATION OF SCIENTIFIC SCHOOLBOOK CONSTRUCTION TO ENTRANCE REQUIREMENTS OF PRIVATE COLLEGES

No book can satisfy the criteria of social worth and thinking outcomes and at the same time satisfy the entrance requirements of a small group of private colleges. Fundamentals of High School Mathematics does not attempt to do this. The entrance requirements for a small group of private colleges must not be permitted longer to thwart the sound construction of public secondary-school textbooks. For any school which wishes to use *Fundamentals of High School Mathematics*, and also prepare for college entrance, the authors are issuing a supplementary pamphlet which will contain *material needed to do the latter but which will be of little or no service in further college work or in mathematical work outside the school.* It is so designed as either to fit into the regular course at various places or to be used as a review and elaboration of the formal material that is now included in the ninth-grade course.

THE TEACHER'S TESTS FOR A TEXTBOOK

Textbooks should be selected for use in our schools, which will satisfy the following tests:

(1) Does the subject matter presented in the book sufficiently justify itself from the point of view of its use or importance either in later school courses, in situations outside the school, or in the thinking outcomes obtained from its study?

(2) Is the subject matter presented in the textbook organized in terms of the ways in which children naturally learn; that is,

has the psychological criterion been kept constantly and adequately in mind?

(3) Is the pupil who takes the course permitted to do real and genuine thinking? Does he have ample opportunity for practice in "problem solving," that is, is the subject matter of the course organized primarily around a core of "problem solving" situations?

The application of these criteria in the construction and selection of school textbooks goes far to bring about the type of reconstruction for which the writers' investigation shows there is a real demand.

HAROLD O. RUGG
JOHN R. CLARK

CHICAGO, ILLINOIS
August 2, 1919

THOSE WHO CONTRIBUTED ESPECIALLY TO THE MAKING OF THIS BOOK

No schoolbook can be made to fit class needs without the coöperation of many teachers and administrators. The superintendents, principals, and selected teachers of 62 high schools made it possible, by their progressive interest and hearty coöperation, to fit the material of the course to the practical needs of the typical "first-year" mathematics class.

Many suggestions for improvement and practical adjustment of the material were made by Mr. J. A. Foberg, Crane High School and Junior College, Chicago; Mr. William Betz, East High School, Rochester, New York; Mr. L. E. Mensenkamp, High School, Freeport, Illinois; Miss Lillian Barnes, High School, Desplaines, Illinois; Miss Florence Morgan, High School, Highland Park, Illinois; Miss Flora E. Balch, Township High School, Evanston, Illinois. These teachers gave of their insight and energy unsparingly, and their contribution has been very large.

The initial classroom experimentation of the authors would have been impossible without the coöperation of Mr. C. W. French, Principal of the Parker High School, Chicago. By his interest in the work and his help in arranging class programs, the authors' paired-teacher plan of experimental teaching was made possible.

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FUNDAMENTALS OF HIGH SCHOOL MATHEMATICS

CHAPTER I

HOW TO USE LETTERS TO REPRESENT NUMBERS IN SOLVING PROBLEMS

Section 1. It saves time to use abbreviations and letters, instead of words, to represent numbers. In order to save time in reading and writing numbers in your studies in arithmetic, you have already found it convenient to use certain **abbreviations** or **letters** to represent numbers. For example, instead of "*dozen*" you have used "*doz.*" to stand for 12; C to stand for 100; M for 1000; cwt. (hundredweight) for 100 lb.; mo. (month) for 30 days, etc. It is necessary that we learn more about this **new way of representing numbers by letters** because we shall use it in all our later work in mathematics.

EXERCISE 1

PRACTICE IN USING ABBREVIATIONS AND LETTERS TO REPRESENT NUMBERS

1. How many eggs are 6 doz. eggs and 2 doz. eggs?
2. How many days in 3 mo. and $2\frac{1}{2}$ mo.?
3. Change 5 yr. and 2 mo. to mo.
4. If 1 ft. = 12 in., change 4 ft. + 5 in. to in.
5. If R (ream) stands for 500 sheets of paper, how many sheets in $2R + 3R$?
6. Change 5 yd. − 2 ft. − 3 in. to in.

7. Using d for 12, how many eggs in $2d + 3d$ eggs?

8. How many sheets of paper in $5R + 3R - 6R$ sheets?

9. Change $5m + 4m - 6m + m$ to smaller time units, that is, to "days," if $m = 30$ days.

10. Change $7y$ (yards) $+ 6f$ (feet) to smaller units, that is, to "inches."

11. If $y = 3f$, how many f in $4y + 6y - 2y$?

12. If h (hour) equals 60 m (minutes), and m equals 60 s (seconds), how many s in $2h + 3m$?

13. If $c = 100$, what is the value of $2c + 4c + c$?

14. A printer uses M for 1000. How many envelopes are there in $5\text{ M} + 3\text{ M} - \text{M}$? What would they cost at \$1.50 per M?

15. How many cents in $4q + 6d$, if q and d stand for the number of cents in a quarter and dime respectively?

16. If $x = 8$, what is the value of $5x + 3x + 2x - 4x$?

In these examples, you have used **abbreviations** or **single letters** to represent numbers or known quantities. We make use of abbreviations or single letters instead of ___?___ because ______________________________. You
State the reason here
need a great deal of practice in doing this. The next exercise will give more practice in representing numbers by letters in different kinds of examples.

EXERCISE 2

FURTHER PRACTICE IN USING LETTERS FOR NUMBERS

1. Change $10y + 4f$ to inches (smaller UNITS) if y and f stand for the number of inches in a yard and in a foot, respectively.

2. Express $2h + 5m$ in seconds.

3. What will $4R + 3R + R$ sheets of paper cost at $\frac{1}{2}$¢ per sheet?

4. At 4¢ each, what will $7d - 2d$ eggs cost?

5. The **length** of the rectangle in Fig. 1 is represented by the expression $3f$, and the **width** by the expression $2f$. What expression will represent the **perimeter**? How many inches in the **perimeter** if $f = 12$ inches?

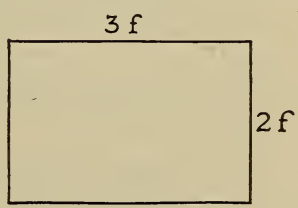

FIG. 1

6. Change $7p + 5p - 8p + 3p$ to ounces, if p stands for the number of ounces in one pound.

7. The expression $14y + 8m + 5d$ represents the age of a pupil in your class. Express this pupil's age in *days* or as *a certain number of d.*

8. Find the cost of $4\frac{1}{2}$ T of coal at 30 cents per *cwt.*, using the **relation**, $T = 20$ *cwt.*

9. If $d = 4q$, find how many q in $3d + 5d$.

10. If $y = 12m$, and $m = 30d$, how many d in $2y + 4m$?

11. A boy earned 27 dollars in a month; his father earned n dollars. How many dollars would both earn in 3 months, if n stands for 60 dollars?

12. What is the numerical value of $4b$ when b is 5? When b is 8?

13. The side of a square is represented as s inches long. What will represent its perimeter?

14. If b represents the number of feet in the base of a rectangle, and h represents the number of feet in its height, what will represent its perimeter? What will represent its area?

15. If pencils cost c cents each, what will n pencils cost?

16. What will represent the cost of one tablet if n tablets cost c cents?

17. If n represents a boy's age, what will represent his father's age, if the father is 25 years older than the son?

18. The width of a rectangle is represented by w. If its length is 6 inches longer than its width, what will represent the length? What expression will stand for the perimeter?

THE PRACTICAL USE OF LETTERS FOR NUMBERS IN FORMULAS

Section 2. Further need for abbreviated language: Shorthand rules of computation. People who have found it necessary to compute over and over again the *areas* or *perimeters* of such figures as rectangles, triangles, circles, etc., have found it very convenient to abbreviate the rules for solving these problems into a kind of *shorthand expres-*

sion which can be more easily written or spoken than the long rules. For example, suppose you wanted to make a complete statement, either in writing or orally, concerning how to find the area of the rectangle which is represented by Fig. 2. You might express it, as you did in arithmetic, as follows:

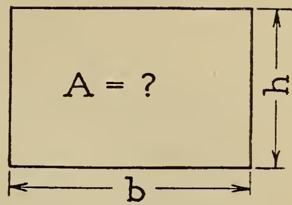

FIG. 2

> **(1) The number of square units in the area of a rectangle is the number of units in its base times the number of units in its height.**

This long word rule can be greatly shortened by using abbreviations or suggestive letters to represent the number of units in each of its dimensions. Thus, a shorter way of expressing this rule is:

$$\text{(2)}\quad \textbf{Area} = \textbf{base} \times \textbf{height}.$$

A third and still more abbreviated way of expressing it is:

$$\text{(3)}\quad A = b \times h,$$

in which A, b, and h mean, respectively, the **number of** UNITS in the area, base, and height. And finally, remembering that $b \times h$ is usually written as bh, the entire statement becomes:

$$\text{(4)}\quad A = bh.$$

This last statement tells us everything that the first statement did, and requires much less time to read or to write. Such algebraic expressions are called FORMULAS.

Section 3. What is a formula? From the previous illustration we see that a **formula** is a shorthand, abbreviated rule for computing. We must remember, however, that the **formula** $A = bh$ is, at the same time, **an equation.**

HOW TO COMPUTE AREAS AND PERIMETERS BY MEANS OF FORMULAS

EXERCISE 3

COMPUTATION OF THE AREA OF RECTANGLES BY THE FORMULA

1. **Illustrative example.** Find the area of a rectangle (Fig. 3) in which $b = 10$ and $h = 7.5$, using the *formula*

$$A = bh.$$

Solution: (1) $A = bh.$
(2) $A = 10 \times 7.5.$
(3) $A = 75.$

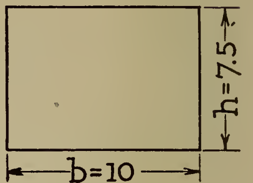

FIG. 3

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2. Find A when $b = 8.25$ and $h = 4$.
3. Find A when $h = 4.5$ and $b = 12$.
4. Find b when $A = 50$ and $h = 5$.
5. What is A if $b = 6.5$ and $h = 5.4$?
6. What is h if $A = 40$ and $b = 6\frac{2}{3}$?
7. Find A if $h = 2.5$ and $b = 6.4$.
8. What is b if $A = 450$ and $h = 22.5$?
9. If $A = 200$ and $b = 7.5$, what does h equal?
10. If $A = 625$ and $h = 50$, what does b equal?
11. What is A if $b = 40$ and h is twice as large as b?
12. Find A if $h = 16.2$ and $b = \frac{1}{2}$ of h.
13. $b = 12$ and $h = \frac{2}{3}b$. What is A?
14. Find A if $h = 20$ and $h + b = 32$.
15. Write a formula for finding the base of any rectangle.

Section 4. Perimeters of rectangles. In Section 3 we saw that it was convenient to use a **formula** for the **area**

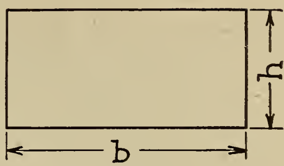

FIG. 4

of a rectangle. In the same way it is helpful to have a **formula** for the **perimeter** of any rectangle.

Since the perimeter of any rectangle is the sum of the bases and altitudes, the shortest way to express this is:

(1) Perimeter = 2 × base plus 2 × height,
or by the **formula**

(2) $P = 2b + 2h$.

EXERCISE 4

COMPUTATION OF THE PERIMETERS OF RECTANGLES BY THE FORMULA

1. **Illustrative example. Find the perimeter if the base is 13 and the height is 9;
or, more briefly, find P if $b = 13$ and $h = 9$.**

Solution: (1) $\mathbf{P = 2b + 2h}$.
(2) $\mathbf{P = 2 \cdot 13 + 2 \cdot 9}$.
(3) $\mathbf{P = 26 + 18 = 44}$.

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2. Find P if $h = 10.5$ and $b = 9$.
3. Find P if $h = 18.4$ and $b = 12.8$.
4. What is h if $P = 40$ and $b = 10$?
5. What is b if $P = 60$ and $h = 14$?
6. Find h if $P = 18.4$ and $b = 4.6$.
7. If $P = 110$ and $h = 22.5$, what is b?
8. What is P if $h = 18$ and $b = 2h$?
9. $P = 100$. Find b and h if $b = h$.
10. What is h if $P = 120$ and $b = \frac{1}{5}P$?

Section 5. The formula for the area of any triangle. What is the area of this triangle if its base is 12 ft. and its

height is 8 ft.? How do you find the area of any triangle if you know its base and altitude?

Show that the most economical way to state this rule, or RELATION, between the area, the base, and the height, is by the formula $A = \frac{bh}{2}$.

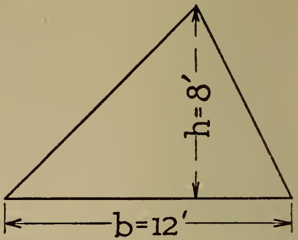

FIG. 5

The examples in the following exercise will give you practice in using this important **formula**.

EXERCISE 5

COMPUTATION OF THE AREA OF TRIANGLES BY THE FORMULA

1. **Illustrative example.**

Find the value of A if $b = 22$ and $h = 12$.

Solution: (1) $A = \frac{bh}{2}$.

(2) $A = \frac{22 \times 12}{2} = \frac{264}{2} = 132$.

Write your work in a neat, systematic form.

2. Find the value of A if $b = 18$ and $h = 6\frac{1}{2}$.
3. What is the value of A if $b = 12.5$ and $h = 20$?
4. If $h = 16.8$ and $b = 28$, what does A equal?
5. What is the value of b if $A = 300$ and $h = 50$?
6. Find h if $A = 240$ and $b = 20$.
7. Determine b if $A = 100$ and $h = 15$.
8. What is A if $h = 6.25$ and $b = 10.5$?
9. Find the value of A if $b = 22$ and $h = \frac{6}{11} b$.
10. What is b if $A = 120$ and $h = \frac{1}{5} A$?
11. $h = 20$ and $b = \frac{7}{5} h$. What does A equal?

12. Can you find b and h if $A = 100$ and $b = 2h$?

13. Two triangles have equal bases, 10 in. each, but the height or altitude of one is twice that of the other. Are their areas equal? Show this by an illustration.

14. What change occurs to A if b is fixed in value, but if h gets larger? What is the RELATION between A and h if b is fixed?

HOW TO USE LETTERS IN TRANSLATING WORD STATEMENTS INTO ALGEBRAIC STATEMENTS

Section 6. Word statements about quantities may be much more briefly expressed by using a single letter to represent a quantity. In the first section we saw that it saved time to use abbreviations or letters to make short-hand rules of computation. Now we shall show that **entire word statements** about quantities may be expressed much more briefly **by using a single letter to represent a number.** To illustrate, consider next the four different ways of writing the statement of the same example.

Illustrative example.

(*a*) The "word" method of stating the example.	(*a*) There is a certain number such that if you add 5 to it the result will be 18. What is the number?
(*b*) An abbreviated way to write it.	(*b*) What no. plus 5 equals 18?
(*c*) A more abbreviated way to write it.	(*c*) No. $+ 5 = 18$.
(*d*) The best way to write it.	(*d*) $n + 5 = 18$.

It is clear that in all these cases the number is 13, and that it is most easily represented by the single letter n.

Thus, the fourth method, $n + 5 = 18$, illustrates the very great saving that is obtained by the use of *single* letters for numbers. This method will be used throughout all our later work. One of the aims of this course is to help you solve problems by better methods than you knew in arithmetic.

EXERCISE 6

Express the following word statements in the **briefest possible** way, **using a single letter** to represent the quantity you are trying to find.

Illustrative example. Write, as briefly as possible: A certain number increased by 7 gives as a result 16.

This may be most briefly expressed:

$$n + 7 = 16,$$
$$\text{or, } n = 9.$$

1. There is a certain number such that if you add 12 to it, the result will be 27. What is the number?
2. If John had 7 more marbles, he would have 18. How many has he?
3. If the length of a rectangle were 5 inches less, it would be 21 inches long. What is its length?
4. A certain number increased by 12 gives as a sum 35. What is the number?
5. The sum of a certain number and 7 is 18. Find the number.
6. Three times a certain number is 21. Find the number.

Explanation: Again, to save time, we agree that 3 times n (8 times p, or 12 times x, etc.) shall be written $3 \cdot n$ or, more briefly, $3n$. Understand,

therefore, that whenever you meet expressions like $8p$, $12x$, $17y$, etc., they mean multiplication, even though no "times" sign ($\times$) is printed. In the same way $\frac{2}{3}$ of a certain number is written $\frac{2}{3}n$ or $\frac{2n}{3}$; $\frac{1}{2}$ of a certain number, $\frac{n}{2}$ or $\frac{1}{2}n$.

7. Two thirds of a certain number is 10. Find the number.

8. Three fourths of a certain number is 15. Find the number.

9. Three fifths of a certain number is 27. What is the number?

10. The difference between a certain number and 5 is 9. What is the number?

"The difference between two numbers" means "the first number minus the second number."

11. The difference between a certain number and 12 is 13. What is the number?

12. The sum of 16 and a certain number is 29. Find the number.

13. The product of 11 and a certain number is 77. Find the number.

14. If your teacher had \$50 less, he would have \$15. How much has he?

15. The quotient of a number and 7 is 3. What is the number?

16. If the area of a rectangle were increased 12 sq. ft., it would contain 40 sq. ft. What is its area?

17. 13 exceeds a certain number by 4. What is the number?

    Explanation: Does this mean that 13 is larger, or smaller, than the certain number? How do you determine how much larger one number is than another?

18. Two thirds of the number of pupils in a class is 28. How large is the class?

19. Tom lacked $7 of having enough to buy a $50 Liberty Bond. How much did he have?

20. Three times a certain number plus twice the same number is 90. Find the number.

21. The difference between 20 and a certain number is 4. What is the number?

22. The number of pennies Harry has exceeds 30 by 7. How many has he?

23. The sum of two numbers is 40. One of them is 27. What is the other?

24. The difference between two numbers is 21. The larger is 60. What is the smaller?

25. The product of two numbers is 95. One is 5. What is the other?

26. The quotient of two numbers is 13. The divisor is 5. Find the dividend.

In these exercises you have been changing, or **translating**, from the language of **ordinary words** into **algebraic language**; you have been making **algebraic** statements out of word statements. The essential thing in this translation is **the representation of numbers by letters.** You should note carefully also that you have begun the practice of using a letter to stand for a number which is **unknown.**

HOW TO "SOLVE AN EQUATION"

Section 7. The most important thing in mathematics: the EQUATION. In all the examples in Exercise 6 you have *translated* word sentences into algebraic sentences. These algebraic sentences are called **EQUATIONS.**

They are called **EQUATIONS** because they show that one number expression is equal to another number expression. For example, you have stated that $n+5=18$. This statement merely expresses equality between the number expression $n+5$, on the left side of the $=$ sign, and the number expression 18, on the right side.

Furthermore, you have been *finding the value* of the unknown number in each of these **EQUATIONS.** From now on, instead of saying "*find the value of the unknown in an equation*," we shall say: "**SOLVE THE EQUATION.**" For example, if you **SOLVE THE EQUATION**

$$7s+s=32,$$

you "find the value" of s; namely, $s=4$.

EXERCISE 7

SOLVE the following **EQUATIONS**:

1. $p+3=8$. This might be written: What no. $+3=8$, or $?+3=8$.
2. $x-5=10$. This might be written: What no. $-5=10$, or $?-5=10$.
3. $2n=25$. This might be written: 2 times $?=25$.
4. $5a=275$. This might be written: 5 times $?=275$.
5. $\frac{1}{2}x=7$. This might be written: $\frac{1}{2}$ times $?=7$.

6. $\frac{2}{3}c = 12$. This might be written: $\frac{2}{3}$ times $? = 12$.

It is always helpful to think of an equation as asking a question. Thus, $5a + 1 = 16$ should be thought of as the question: 5 times what number plus one gives 16?

7. $2b + 1 = 21$
8. $5d - 3 = 27$
9. $4x = 13$
10. $12 = 3p$
11. $5 + n = 11$
12. $6 - n = 2$
13. $2p + 3p = 35$
14. $3x + 1 = 16$
15. $7b - 2 = 12$
16. $12s = 27$
17. $16 = 5y - 1$
18. $6t + 3t = 27$
19. $13 = 5y$
20. $21 = 5x + 1$
21. $4y + 3 = 20$
22. $6 + 2x = 14$
23. $27 = 5y + 2$
24. $\frac{2}{3}n = 18$
25. $2b - 5 = 11$
26. $4x + 5x = 18$
27. $3y - 8 = 19$
28. $10y = 25$
29. $5a - 1 = 14$
30. $x + x = 18$
31. $\frac{3}{5}y = 15$
32. $\frac{1}{2}n + \frac{1}{2}n = 20$
33. $\frac{3}{4}n + \frac{1}{4}n = 30$
34. $\frac{2}{5}n + \frac{1}{5}n = 18$

HOW TO REPRESENT TWO OR MORE UNKNOWN NUMBERS, WHEN THEY HAVE A DEFINITELY KNOWN RELATION TO EACH OTHER

Section 8. In the examples which you have worked in preceding lessons, you have had to represent **only one number** in each problem. To illustrate, in Example 9, Exercise 6, as in all the other examples solved thus far — "the difference between a certain number and 5 is 9." Only ONE number has to be represented. But in most

of the examples that you will meet in mathematics you will have to represent two or more numbers which have a definitely known RELATION to each other. For example, consider this problem:

> **Suppose Tom has 5 times as many marbles as John has. How many do they both have?**

It is clear that there are *two* numbers to be represented; namely, the number that Tom has and the number that John has. Furthermore, since there is a definite RELATION between these two numbers, that is, one is 5 times the other, it is important to see that **each** can be represented by the use of the **same letter.**

If you **let** n stand for the number John has, what **must** represent the number Tom has? Since the example states that Tom has 5 times as many as John, **then** Tom must have $5n$ marbles. In the same way, together they have the sum of the two; namely, $n + 5n$, or $6n$.

The best way to state this, however, *in algebraic language*, is to use a set form like the following:

Let n = the number John has.
Then $5n$ = the number Tom has,
and $5n + n$, or $6n$ = the number both have.

The next exercises will show how two or more unknown numbers may be represented by using the same letter, if the numbers have a **definite relation** to each other.

EXERCISE 8

1. Harry has four times as many dollars as James has. If you let n stand for the *number* of dollars James has, what expression will stand for

the *number* Harry has? for the number they together have?

2. The number of inches in the length of a rectangle is 7 times the number in its width. If n stands for the *number* of inches in its width, what will represent the number in its length? in its perimeter?

3. An agent sold three times as many books on Wednesday as he sold on Tuesday. *Represent* the number sold each day. State algebraically that he sold 28 books during both days.

4. There are twice as many boys as girls in a certain algebra class. If there are n girls, how many boys are there? How many pupils? State algebraically that there were 36 pupils in the class. Find the number of boys.

5. On a certain day Fred sold half as many papers as his older brother. How can you represent the number each sold? the number both sold?

6. During a certain vacation period there were three times as many cloudy days as clear days. Express the number of each kind of days, and the total number of days. If the vacation consisted of 60 days, how many days of each kind were there?

7. A rectangle is three times as long as it is wide. If it is x feet wide, how long is it? What is its perimeter?

8. If one side of a square is s inches long, what is the perimeter of the square? State algebraically that the perimeter is 108 inches.

9. The sum of three numbers is 60. The first is three times the third, and the second is twice the third. If n represents the third number, what will represent the first? the second? their sum? State algebraically that the sum is 60, and then find each number. Why do you think it was advisable to represent the **third** number by n?

10. John sold five times as many papers as Eugene. If n represents the number Eugene sold, what will represent the number John sold? What expression will represent the difference in the number sold? Make a statement showing that this expression is 32.

11. A farmer sold four times as many dollars' worth of wheat as of corn. If he received x dollars for the corn, what will represent the amount he received for both?

12. A has n dollars. B has three times as many as A, and C has as many as both A and B. What will represent the number of dollars all three together have?

13. A horse, carriage, and harness cost $500. The carriage cost three times as much as the harness, and the horse twice as much as the carriage. If you let n represent the number of dollars the harness cost, what will represent the cost of the carriage? of the horse? of all together? Make an algebraic statement showing that all three cost $500. Can you now find the cost of each?

14. A man had 400 acres of corn and wheat, there being 7 times as much corn as wheat. Show how the number of acres of each could be represented by some letter. Make an algebraic statement showing that he had 400 acres of both.

15. The rectangle shown in Fig. 6 is three times as long as wide. State algebraically that the perimeter is 64 in. What are its dimensions?

FIG. 6

In the problems just studied you have been considering two or more numbers which had a **definite relation** to each other and all of which *had* to be represented by using the **same letter.** For example, you had to note that one number was always a certain number of times another one, or was a certain part of another one. In each problem you had to decide which of the unknown numbers you would represent by that letter. In general it is best to represent the **SMALLEST** of the unknown numbers by n or by p or by **ANY** letter. *The other numbers must then be represented by using* **the same letter** which you selected to represent the first one.

EXERCISE 9

Write out the solution of each of the following. Be sure to **use the complete form illustrated below,** in solving each example.

Illustrative example. **The larger of two numbers is 7 times the smaller. Find each if their sum is 32.**

Let $s =$ smaller no.
Then $7s =$ larger no.
and $7s + s = 32$,
or $8s = 32$,
or $s = 4$, the smaller number,
and $7s = 28$, the larger number.

1. William and Mary tended a garden, from which they cleared \$72. What did each receive if it was agreed that William should get three times as much as Mary?

2. The perimeter of a rectangle is 48 inches. Find the dimensions if the length is 5 times the width.

3. The sum of three numbers is 60. The second is twice the first, and the third equals the sum of the first and second. Find each.

4. Divide \$48 between two boys so that one shall get three times as much as the other.

5. Twice a certain number exceeds 19 by 5. Find the number.

6. The product of a certain number and 5 is 35. Find the number.

7. A man is twice as old as his son. The sum of their ages is 90 years. Find the age of each.

8. The sum of three numbers is 120. The second is twice the first and the third is three times the first. Find each.

9. The perimeter of a certain square is 144 inches. Find the length of each side.

10. The perimeter of a rectangle is 160 inches. It is three times as long as it is wide. Find its dimensions.

11. William is three times as old as his brother. The sum of their ages is 36 years. How old is each?

12. One number is five times another. Their difference is 16. Find each.

13. The sum of three numbers is 14. The second is twice the first, and the third is twice the second. Find each number.

14. One number is eight times another. Their difference is 63. Find each.

15. A rectangle (Fig. 7) which is formed by placing two equal squares together has a perimeter of 150 feet. Find the side of each square, and the area of the rectangle.

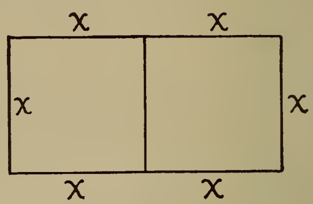

FIG. 7

16. Three men, A, B, and C, own 960 acres of land. B owns three times as many acres as A, and C owns half as many as A and B together. How many acres has each?

17. John sold half as many thrift stamps as Harry sold; Tom sold as many as both the other boys together. Find how many each sold, if all sold 144 thrift stamps.

TRANSLATION FROM ALGEBRAIC EXPRESSIONS INTO WORD EXPRESSIONS

Section 9. In the previous work you have translated from word statements into algebraic expressions. It is also **very helpful to translate the algebraic expressions back into word expressions.** For example, $n+4=13$ is the same as the words tatement "*the sum of a certain number and* 4 *is* 13." In the same way, the algebraic statement $4y=26$ should be translated as follows:

"the product of a certain number and 4 is 26," or
"four times a certain number equals 26."

The next exercise will give practice in this important process, *i.e.* translating from algebraic statements into word statements.

EXERCISE 10

TRANSLATE EACH OF THE FOLLOWING ALGEBRAIC STATEMENTS INTO WORD STATEMENTS

1. $y+4=20$
2. $2b+1=31$
3. $13=2+y$
4. $2a+3a=55$
5. $\frac{1}{2}n+1=12$
6. $n+3n=24$
7. $\frac{n}{5}+1=8$
8. $\frac{1}{2}x+3=8$
9. $5x=18$
10. $\frac{18}{g}=3$
11. $\frac{2y}{5}=8$
12. $c-4=12$
13. $20-n=12$
14. $\frac{p}{2}+\frac{p}{3}=10$

SUMMARY OF CHAPTER I

After studying this chapter you should have clearly in mind:

1. It saves time to *represent numbers by letters.*
2. It is very economical to abbreviate commonly used rules of computation into formulas which consist altogether of letters.
3. Worded problems may be translated into *algebraic statements.*
4. *Equations* are statements that two numbers or two algebraic expressions are equal.
5. *Solving equations* means finding the value of the unknown number or letter in the equation.
6. *Algebraic expressions* may be *translated into word expressions.*

CHAPTER II

HOW TO USE THE EQUATION

Section 10. The importance of the equation. Nothing else in mathematics is as important as the *equation*, and the power to use it well. It is a *tool* which people use in stating and solving problems in which an unknown quantity must be found. In the last chapter we saw that the *formula*, or *equation*, was used to find unknown quantities, sometimes the area, sometimes the perimeter, etc. The fact that the **equation** is used as a means of solving such a large number of problems is the reason we shall study it very thoroughly in this chapter.

Section 11. The equation expresses balance of numerical values. The equation is used in mathematics for the same purpose that the weighing "scale" is used by clerks; that is, to help in finding some value which is unknown. The **scale represents balance of weights**; similarly the **equation represents balance of numerical values.** To understand clearly the principles which are applied in dealing

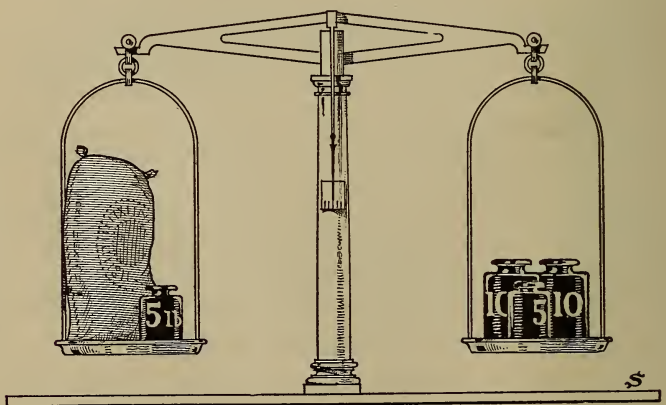

FIG. 8

with equations, we should consider the scale, as represented in Fig. 8. In this case a bag of flour of unknown weight, together with a 5-pound weight, balances weights which total 25 pounds on the other side of the scale.

Now, if n represents the number of pounds of flour, it is clear that the **equation**

$$n + 5 = 25$$

represents a balance of numerical values. Obviously, n is 20, for the clerk would take 5 pounds of weight from each side, and still keep a **balance of weights.**

This principle, namely, that the same weight may be taken from each side without destroying the balance of weights, can be applied to the equation

$$n + 5 = 25.$$

That is, we may **subtract 5 from each side** of the equation, giving another equation,

$$n = 20.$$

This suggests an important principle that may be used in solving equations; namely, —

> **The same number may be subtracted from each side of the equation without destroying the equality, or balance, of values.**

If you take something from one side of the scale, or of the equation, what MUST you do to the other side? **Why?**

The fact that the equation expresses the idea of balance makes it easy to reason about it, and find out all the things that can be done without changing the balance or equality. The next exercise suggests this kind of study of the equation.

EXERCISE 11

By **thinking of the equation as a balance**, you should be able to complete the following statements. **Fill in the blanks with the proper words.**

1. Any number may be subtracted from one side of an equation if ___?___ is ___?___ from the other side.
2. Any number may be added to one side of an equation if ___?___ is ___?___ to the other.
3. One side of the equation may be multiplied by any number, if the other side is ___?___ by the ___?___.
4. One side of an equation may be divided by any number, if the other side is ___?___ by the ___?___.

These are very important principles, and are used in solving any equation. They are generally called AXIOMS. They must be understood and mastered. The examples of the next exercise have been planned to help you learn how to apply them.

EXERCISE 12

In each of the following examples you can use one of the four principles stated above to explain what has been done, or to state the reason for doing it. Thus, if $2x = 8$, then $x = 4$, because of the principle:

"One side of an equation may be divided by a number if the other side is divided by the same number."

For each example, you are to **state the principle which permits or justifies the conclusion.**

1. If $4b = 22$, then what is done to each side to give $b = 5\frac{1}{2}$?
2. If $\frac{1}{2}y = 7$, then to get $y = 14$, what do you do to each side?
3. If $x + 4 = 13$, then to get $x = 9$, what do you do to each side?
4. If $5c = 32.5$, then what is done to each side to give $c = 6.5$?
5. If $6a = 12$, then what is done to each side to give $3a = 6$?
6. If $y - 4 = 7$, then what is done to each side to give $y = 11$?

   Hint: Which expression is the larger, y or $y - 4$? How much larger? Then what must be done to the smaller expression to make it equal to the larger expression?

7. If $x - 6 = 10$, what is done to each side to get $x = 16$?
8. It is known that $y - 5 = 7$. Why does $y = 12$?
9. What is done to $n - 6 = 11$ to get $n = 17$?
10. If $a - 1 = 9$, then why does $a = 10$?
11. If $b = 2h$, then what is done to each side to give $3b = 6h$?
12. If $4x + 3 = 23$, then what is done to each side to give $4x = 20$?
13. If $5y - 3 = 27$, then what is done to each side to give $5y = 30$?
14. If $x + 7 = 19$, then to make $x = 12$, what is done to each side?
15. If $2c - 4 = 8$, then to make $2c = 12$, what is done to each side?

16. If $3b + 1 = 22$, then what is done to each side to give $b = 7$?

17. If $5b + 2 = 47$, why does $5b = 45$? Then why does $b = 9$?

18. If $6x + 2 = x + 22$, then why does $5x + 2 = 22$? then why does $5x = 20$? then why does $x = 4$?

19. If you know that $4w + 3 = w + 27$, then why does $4w = w + 24$? then why does $3w = 24$? then why does $w = 8$?

These examples are given to emphasize the fact that there are certain changes that can be made on both sides of an equation, without destroying the balance or equality. It should be clear that **there must be some axiom or principle to justify every change that is made.**

EXERCISE 13

Find the value of the unknown number in each of the following equations, *telling exactly what you do to each side of the equation.*

1. $x + 5 = 13$
2. $3a = 17$
3. $26 = 4y$
4. $2b + 1 = 19$
5. $y - 5 = 12$
6. $\frac{1}{2}a = 4.5$
7. $\frac{a}{3} = 5$
8. $2x - 3 = 17$
9. $15 = x + 7$
10. $2x + 3x = 35$
11. $5y + 4y + y = 30$
12. $5c - 2 = 38$
13. $27 = 6x - 3$
14. $4b + 7 = 47$
15. $4y - y = 21$
16. $5x + 1 = 23$
17. $2\frac{1}{2}x = 15$
18. $\frac{2}{3}a = 18$
19. $2b + 3b = 42$

20. $c - 4 = 13$

21. $2b - 1 = 18$

22. $18 + x = 13 + 10$

23. $7 + 2x = 23 + 10$

24. $\frac{1}{2}x + \frac{1}{4}x = 18$

HOW TO CHECK THE ACCURACY OF THE SOLUTION OF AN EQUATION

Section 12. When is an equation solved? We have already noted that *an equation is solved when the* **numerical value** *of the unknown number is found.* Thus, the equation $4a + 3 = 29$ is **solved** when the numerical value of a is found which makes both sides equal. This leads to another very important question; that is: How can you be certain your solution is correct? In other words, how can you **test** or **check** the accuracy of your work?

For example, suppose that in solving the equation

$$4a + 3 = 29$$

one member of your class obtains 8 for the value of a. Is his result correct? There is only one way to be sure. That is to **substitute** or "put in" 8 in place of a in the equation, to see whether the numerical value of the left side equals the numerical value of the right side. In other words, does

$$4 \cdot 8 + 3 = 29?$$

Clearly, not. Therefore, the solution is incorrect; it does not **check**. Then what is the correct value of a? Some of you doubtless think it is $6\frac{1}{2}$. Let us **test** or **check** by substituting $6\frac{1}{2}$ for a, to see if the numerical value of one side of the equation will equal the numerical value of the other side. Does

$$4 \cdot 6\tfrac{1}{2} + 3 = 29?$$

Yes. Then **the equation is solved**, or, to use the more general term, **the equation is** SATISFIED when $a = 6\frac{1}{2}$.

Summing up, then, **an equation is SOLVED when a value of the unknown is found which SATISFIES the equation**; that is, one which makes the numerical value of one side equal to the numerical value of the other side. The *solution* of the equation *is checked*, or proved, by substituting for the unknown number the value which we *think* it has. If, as the result of the substitution, we get a balance of values, then we know that the equation has been solved correctly.

EXERCISE 14

PRACTICE IN CHECKING THE SOLUTION OF EQUATIONS

1. The pupils in a class tried to solve the equation

$$6a - 3 = 39.$$

A few decided that $a = 7$, while the others insisted that $a = 6$. Which group was right? Show how they could have checked or tested their result. Why, do you think, did some pupils get 6 for the value of a?

2. Does $x = 5$ in the equation $12x - 7 = 10x + 3$? In other words, does $x = 5$ **satisfy** this equation?

3. Would you give full credit on an examination to a pupil who said that $y = 4\frac{1}{2}$ would **satisfy** the equation $8y - 4 = 6y + 3$? Justify your answer.

4. Show whether the equation $b^2 + 5b = 24$ is **satisfied** or solved if $b = 3$; if $b = 2$.

5. Do you agree that the value of x is 6 in the equation $10x - 4 = 58$? Justify your answer.

6. Is the equation $\dfrac{b+10}{3} + 6 = \dfrac{3}{4}b + 8$ **satisfied** when $b = 8$?

7. Does $x = 24$ **satisfy** the equation

$$\tfrac{1}{2}x + \tfrac{1}{3}x + \tfrac{3}{8}x = 29?$$

8. State in words how the solution of an equation is tested or checked.
9. What is the value of learning to check very carefully every kind of work you do?

EXERCISE 15

Solve each of these equations. **Write out your work for each one** in the complete form illustrated in the first example. **Check each one** so that you can be absolutely certain that your work is correct.

1. **Illustrative example.**

$$6b - 4 = 24.$$

(1) By adding 4 to each side, we get

$$6b = 28.$$

(2) By dividing each side by 6, we get

$$b = \tfrac{14}{3}.$$

(3) Checking,

$$6 \cdot \tfrac{14}{3} - 4 = 24.$$

$$28 - 4 = 24.$$

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2. $5c - 2 = 38$
3. $6b + 3 = 45$
4. $7x = x + 30$
5. $10a - 3a = 17\frac{1}{2}$
6. $22 = 5x + 2$
7. $2\frac{1}{2}y + 1 = 26$
8. $b + 5b = 20 + b$
9. $4y = 13 + y$
10. $3x + 2x + 6x = 66$
11. $5c + 3 = 78$
12. $\frac{1}{2}y = 8\frac{1}{2}$
13. $\dfrac{n+1}{5} = 4$
14. $10b + 3 = 7b + 15$
15. $12x - 2 = 5x + 26$
16. $13y = 2y + 3y + 4y + 8$
17. $8y + 2 = 3y + 37$
18. $4 + 9x = 2x + 46$
19. $7b - 3 = 2b + 12$
20. $8w - 5 = w + 51$
21. $x + 2x + 3x = 48$
22. $y + 2y + 3y + 1 = 61$
23. $5x - 2 = 34 - x$

HOW TO GET RID OF FRACTIONS IN AN EQUATION

Section 13. The use of the most convenient multiplier.* In many equations that you have solved already it has been necessary to multiply each side of the equation by some number. For example, in $\frac{1}{2}x = 10$, it is necessary to multiply each side by 2, which gives $x = 20$. Or, if you wanted to solve the equation $\frac{1}{5}x = 3$, it is necessary to multiply each side by 5, giving $x = 15$.

But, suppose you had an equation

$$\tfrac{1}{2}x + \tfrac{1}{5}x = 14,$$

would you get rid of *both* fractions by multiplying each side by 2? Would you get rid of *both* fractions by multiplying each side by 5? Here, as in all equations of this kind, you have to find some number which is a multiple of the different denominators. For this reason, in this example, 10 is the **most convenient number** by which to multiply each term in the equation.

Illustrative example.

(1) $\qquad \frac{1}{2}x + \frac{1}{5}x = 14.$

Multiplying each side of (1) by 10, we get

(2) $\qquad 10 \cdot \frac{1}{2}x + 10 \cdot \frac{1}{5}x = 10 \cdot 14,$

(3) or $\qquad 5x + 2x = 140,$

(4) or $\qquad 7x = 140,$

(5) or $\qquad x = 20.$

Checking, by substituting the value of x in the original equation, gives

$$\tfrac{1}{2} \cdot 20 + \tfrac{1}{5} \cdot 20 = 14,$$
$$10 + 4 = 14.$$

The study of this example shows that we can get rid of fractions in an equation if we multiply each side by the

* This helpful phrase was first suggested to the authors by Mr. J. A. Foberg, Crane High School and Junior College, Chicago.

lowest common multiple of the denominators. We shall call this the MOST CONVENIENT MULTIPLIER.

EXERCISE 16

PRACTICE IN SOLVING FRACTIONAL EQUATIONS

Solve and check each example.

1. **Illustrative example.** $\frac{2}{3}n + \frac{1}{4}n = 22$. What is n?

(1) Multiplying each side by the *most convenient multiplier*, 12, gives

$$12 \cdot \tfrac{2}{3}n + 12 \cdot \tfrac{1}{4}n = 12 \cdot 22,$$

or, $$8n + 3n = 264,$$

or, $$11n = 264.$$

(2) By dividing each side by 11, we get

$$n = 24.$$

(3) Checking, by substituting the value of n (24), in the original equation,

$$\tfrac{2}{3} \cdot 24 + \tfrac{1}{4} \cdot 24 = 22.$$
$$16 + 6 = 22.$$

2. What is the value of x in the equation

$$\tfrac{1}{2}x + \tfrac{1}{5}x = 14?$$

3. A man spent $\frac{1}{5}$ of his income for rent and $\frac{1}{6}$ for groceries. Using n to represent his income, make an equation which will state that he spent \$ 660 for rent and groceries. Solve the equation.

4. The dimensions of a rectangle are indicated on Fig. 9. What equation will state that the perimeter is 36 in.? Solve the equation for l.

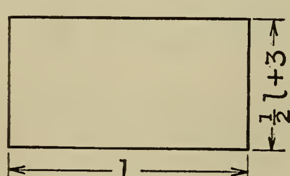

FIG. 9

5. $\frac{4}{5}b + b = \frac{1}{2}b + 13$. What is b?

6. $\frac{2}{3}x + x = \frac{1}{2}x + 14$.

7. If three fourths of a certain number be diminished by one half of the number, the remainder is 10. Find the number.

8. $\frac{2}{3}x + \frac{3}{4}x - 1 = 3$.

9. $\frac{2}{5}y - \frac{1}{6}y = \frac{1}{5}y + 1$.

10. Three boys together had 65 cents. Tom had half as much as Harry, and Bill had two thirds as much as Harry. Translate this into an equation and solve.

11. $\frac{n}{2} + 2 + \frac{n}{3} = 4 + \frac{n}{2}$. What does n equal?

12. $x - \frac{x}{5} - 2 = 14$.

13. One half of a certain number increased by four fifths of the same number gives 52 as a result. Find the number.

14. Harry made two thirds as much money last year selling the Saturday Evening Post as John made; Edward made three fourths as much as John. How much did each boy earn if all together earned $145?

15. $x + \frac{2}{3}x - 6 = 24$.

16. The sum of the third, fourth, and sixth parts of a number is 18. Find the number.

17. $\frac{1}{2}y - 7 = \frac{1}{9}y$.

18. $\frac{2}{3}b = \frac{1}{2}b + 2$.

19. Twelve increased by one half of a certain number gives the same result as fifteen increased by one fifth of the certain number. Find the number.

20. $\frac{1}{4}n - \frac{1}{5}n + \frac{5}{6}n = 53$

21. $\frac{y}{3} = \frac{y}{4} + 2$

22. $\frac{2}{3}p + \frac{3}{4}p = p + 5$

23. $8 = \frac{1}{2}x + 2$

24. $y + \frac{2}{3}y = \frac{1}{7}y + 32$

25. $\frac{2}{3}x - 5 = \frac{3}{9}$

26. $\frac{n}{2} = \frac{15}{6}$

27. $\frac{y}{5} = \frac{7}{2}$

28. $\frac{3}{2}p = \frac{9}{4}$

29. $\frac{11}{4} = \frac{x}{2}$

30. Two thirds of the length of a rectangle is 8 more than its width. Its perimeter is 64 inches. What are its dimensions?

31. Two thirds of a certain number is 16 more than two fifths of the same number. What is the number?

32. $\frac{5}{7} = \frac{2}{8}x$

33. $\frac{2}{5}y = \frac{10}{3}$

34. $\frac{14}{x} = 2$. (Multiply each side by x.)

35. $\frac{16}{y} = 8$

36. $\frac{63}{x} = 7$

37. $\frac{20}{2y} = 5$

38. $\frac{36}{2y} = 4$

39. $\frac{50}{x} = 10$

40. $\frac{60}{5b} = 3$

41. $\frac{144}{3p} = 12$

42. $\frac{100}{3y} = 10$

Section 14. How word problems are solved by equations. It is important to note the principal steps involved in solving word problems. Let us take, as an illustration, Example No. 10 in Exercise 16.

Three boys together had 65 cents. Tom had half as much as Harry, and Bill had two thirds as much as Harry. How much had each?

1. The first important step in solving a word problem is to **get in mind** very clearly **what is known and** to recognize **what is to be found out.** In all problems some things are known and some things are to be determined. Thus, in this problem, we know how much money all the boys have together; and we also know that Tom has half as much as Harry; furthermore, we know that Bill has two thirds as much as Harry. That is, we see that the statement of the amount that Tom and Bill each has **depends upon** the statement of the amount that Harry has.
2. But we do not know how much Harry has. Then, as in all word problems, we represent by some letter, such as n, the number of dollars Harry has. In other words, the **second step** is to get clearly in mind what quantities are unknown, and **to represent one of them by some letter.**
3. Next, **all the parts** or conditions of the problem **should be expressed by using the SAME letter.** Thus, if Harry has n dollars, the number that Tom and Bill each has **should** be represented by using the same letter n, and NOT some other letter. That is, the word statement must be **translated** into an algebraic statement. It is always necessary, and usually difficult, to see that there must be a balance, an

equality, between the parts of the problem. Thus, we must see that Harry's money, n, plus Tom's money, $\frac{1}{2}n$, plus Bill's money, $\frac{2}{3}n$, must **balance**, or equal, 65 cents. This gives the complete algebraic statement:

$$n + \tfrac{1}{2}n + \tfrac{2}{3}n = 65.$$

4. The equation which we have obtained must be **solved**. A value of the unknown must be found which will **SATISFY** the equation. In this case n proves to be 30.

5. Finally, the **ACCURACY** of the result must be **TESTED** by substituting the obtained value of n in the original word statement of the problem, to see if the statement holds true.

The word statement says that Tom has half as much money as Harry. Our solution says Harry has 30 cents. Then Tom has 15 cents. Similarly Bill has two thirds as much, or 20 cents. They together had 65 cents, or 30 cents + 20 cents + 15 cents. Thus **our solution checks with the word statements in the problem.**

EXERCISE 17

Translate into algebraic language, and solve each of the following word statements. **Check each one.**

1. Six more than twice a certain number is equal to 12. Find the number.
2. Four times a certain number is equal to 35 diminished by the number. What is the number?
3. I am thinking of some number. If I treble it, and add 11, my result will be 32. What number have I in mind?

4. If fourteen times a certain number is diminished by 2, the result will be 40. Find the number.

5. What is the value of y in the equation

$$4y + \tfrac{1}{3}y + 2 = 28?$$

6. If seven times a certain number is decreased by 8, the result is the same as if twice the number were increased by 32. Find the number.

7. An algebra cost 12 cents more than a reader. Find the cost of each if both cost $1.64.

8. The sum of the ages of a father and his son is 57 years. What is the age of each if the father is 29 years older than the son?

9. The length of a school desk top exceeds its width by 10 inches; and the perimeter of the top is 84 inches. What are its dimensions?

10. Divide $93 between A, B, and C, so that A gets twice as much as C, and B gets $10 more than C.

11. Should a teacher give James full credit for the solution of the equation

$$4\tfrac{1}{2}x - 7 = 3x + 5$$

if he obtained $x = \frac{17}{2}$? Justify your answer.

12. Make a drawing of a rectangle whose perimeter is represented by the expression $6y + 20$, writing the dimensions on the drawing.

13. The length of a rectangle is 5 inches more than twice its width; its perimeter is 46 inches. What are its dimensions?

14. A school garden was $3\frac{1}{2}$ times as long as wide. To walk around it required 31 steps (27 in. each). Tell *how* to find its width, but do not actually find it.

15. Three men went into business together. A put in $500 more than B, and C put in one fifth as much as B. How much did each put in if they together put in $4900?

16. The smaller of two numbers is 8 more than one fifth of the larger. Find each number if their sum is 56.

17. The perimeter of a triangle is 90. The longest side is twice as long as the shortest, and the other one is three halves as long as the shortest. Find the length of each side.

18. The sum of four numbers is 70. The first one is one half of the third; the second is three fifths of the third; and the fourth one is 5 more than the third. Find each number.

19. A farmer sold a certain number of hogs, n, at $20 apiece. What did they bring him? Then he sold twice as many sheep at $14 apiece. What did the sheep bring him? If all together brought him $576, how many of each did he sell?

20. A man had a square piece of ground fenced in for a garden. He made it 10 feet longer, and 8 feet wider. He then needed 236 feet of fence to inclose it. Find its original size.

SUMMARY OF CHAPTER II

From your study of this chapter, the following principles and methods should be kept clearly in mind:

1. Equations express *balance* of value.
2. If any change is made on one side of the equation, the *same* change must be made on the other side.
3. An equation is solved when a value of the unknown is found which *satisfies* the equation.
4. The accuracy of your solution is *checked* by substituting the value of the unknown in the equation, and noting whether the numerical values of the two sides balance.
5. You can get rid of fractions in an equation by multiplying each side by the lowest common multiple of the denominators; that is, by the *most convenient multiplier.*
6. Five important steps must be mastered in solving word problems:

   (*a*) getting clearly in mind what is known and what is unknown;

   (*b*) representing one of the unknowns by some letter;

   (*c*) expressing in an equation all parts of the problem by using the same letter;

   (*d*) solving the equation;

   (*e*) checking the result by substituting the value obtained for n in the original word statement.

CHAPTER III

HOW TO CONSTRUCT AND USE ALGEBRAIC EXPRESSIONS

Section 15. Meaning of an algebraic expression. We have already seen that letters are often used to represent numbers. Any symbol, such as a letter, or a group of symbols which represents a number, is called an algebraic expression. For example, ab is an algebraic expression; $c+d$ is an algebraic expression; $2g+p+y$ is another; $2x+3y+z$, another, etc.

Section 16. The numerical value of an algebraic expression. If we give numerical values to the single letters, then the whole expression has a numerical value. Thus, if $a=2$, and $b=5$, then the numerical value of the expression ab is 10 (that is, 2×5). If $c=3$, and $d=4$, then the numerical value of the expression $c+d$ is 7. Similarly the numerical value of other expressions such as $2g+p-y$, depends upon the values that we give to g, p, and y. In this chapter we shall learn how to construct and use algebraic expressions.

EXERCISE 18

PRACTICE IN FINDING THE NUMERICAL VALUE OF QUANTITIES IN PRACTICAL FORMULAS

1. In the formula $P=2b+2h$ find the value of P when $b=7.5$ and $h=4.2$.
2. Find the value of A in the formula $A=bh$ when $b=6.4$ and $h=4\frac{1}{5}$.
3. What is the value of A in the formula $A=\frac{bh}{2}$ if $b=5.8$ and $h=4.6$?
4. Determine what the value of V is in the formula $V=lwh$, if $l=12$, $w=10$, and $h=9$.

I. "EVALUATION": HOW TO FIND THE NUMERICAL VALUE OF AN ALGEBRAIC EXPRESSION

Section 17. In the examples which you have just solved we have used the long expression "What is the value of" or "**Find the value of**" in referring to the particular letter which was to be found. Instead of these long expressions we shall now use the single word **EVALUATE.** It means exactly the same thing as the longer expression. Thus to **evaluate** an algebraic expression means to find its numerical value, exactly as in the previous examples. This is done by "putting in" or by **substituting** numerical values for the letters. A few examples will make this clear.

EXERCISE 19

SUBSTITUTING NUMBERS FOR LETTERS IN COMMONLY USED FORMULAS

1. Evaluate $A = \frac{bh}{2}$ if $b = 10$ and $h = 14.6$.
2. Evaluate, or find the value of, P in the expression $P = 2b + 2h$ if $b = 26$ and $h = 12.4$.
3. Evaluate $C = 2\pi R$ if $R = 14$. $\pi = \frac{22}{7}$.
4. Evaluate $V = lwh$ if $l = 10$, $w = 6\frac{1}{2}$, and $h = 5$.
5. Find the value of i in the formula $i = prt$ if $p = \$640$, $r = \frac{5}{100}$, and $t = 4$.
6. Evaluate $c = \frac{E}{R}$ if $E = 110$ and $R = 10.5$.
7. What is h in the algebraic expression $P = 2b + 2h$ if $P = 80$ and $b = 12.8$?

EXERCISE 20

PRACTICE IN FINDING THE NUMERICAL VALUE OF ALGEBRAIC EXPRESSIONS

In each of the following examples, let $a = 2$, $b = 3$, $c = \frac{1}{2}$, $x = 5$, and $y = 0$.

Illustrative example.

Evaluate the expression $3a + 2c - \frac{2x}{3}$.

Putting in place of each letter its numerical value, gives

$$3 \cdot 2 + 2 \cdot \tfrac{1}{2} - \frac{2 \cdot 5}{3}$$

$$\text{or } 6 + 1 - 3\tfrac{1}{3}$$

$$\text{or } 3\tfrac{2}{3}.$$

1. $4a + 5b - 2x$
2. $ab + 6c + y$
3. $3ac + 2bx$
4. $abc + xy$
5. $\dfrac{a + c}{x}$
6. $\dfrac{x + 2y}{20c}$
7. $\dfrac{a}{b} + \dfrac{b}{c}$
8. $\dfrac{y}{x} + abc$

II. THE USE OF EXPONENTS TO INDICATE MULTIPLICATION IN ALGEBRAIC EXPRESSIONS

Section 18. Need of short ways to indicate multiplication. A very large part of our work in mathematics is that of **finding numerical values**. In many of our problems, therefore, we shall need short ways of *indicating* multiplication. For example, in arithmetic, the multiplication of 5×5 is sometimes written as 5^2; or the multiplication of $6 \times 6 \times 6$ as 6^3. In *algebra*, to save time, this notation, or method of indicating multiplication, *is always used.* Thus, instead of writing $b \times b$ or $n \times n \times n$ we will write

b^2 or n^3. This little number that is placed to the right of and above another number tells how many times that number is to be used as a factor. These numbers are called **exponents.** Numbers with exponents are read as follows:

> **$3a^2$ means 3 times a times a, and is read "3 a *square.*" This does NOT mean 3 a times 3 a. The exponent affects only the a.**
>
> **$5b^3$ means 5 times b times b times b, and is read "5 b *cube.*" This does NOT mean 5 b times 5 b times 5 b. The exponent affects only the b.**

Here, as well as throughout all later mathematical work, you will need to be able to evaluate algebraic expressions which involve exponents. For example, the *area* of the rectangle shown here is the expression $3W^2$, which is obtained by multiplying $3W$ by W. Now the *numerical value* of this area

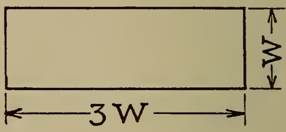

FIG. 10

depends upon *the value of W;* that is, if W is 4, then the area is $3 \cdot 4 \cdot 4$, or 48; but if W is 2, then the area is $3 \cdot 2 \cdot 2$, or 12. In the same way the *volume* of the rectangular box in Fig. 11 is represented by the expression $2x^3$, or $2x \cdot x \cdot x$. Again, you see that the numerical value of the volume **depends upon the value of x.** Thus, if x is 5, the volume is

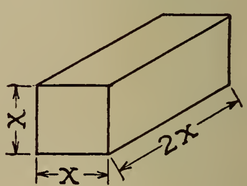

FIG. 11

obtained by **evaluating** the expression $2x^3$, which gives $2 \cdot 5 \cdot 5 \cdot 5$, or 250. The next exercise gives practice in evaluating algebraic expressions which contain exponents.

EXERCISE 21

PRACTICE IN THE EVALUATION OF ALGEBRAIC EXPRESSIONS WHICH CONTAIN EXPONENTS

1. **Illustrative example.**

   Evaluate $2ab^2 + 3a^2b + ac$, if $a = 4$, $b = 3$, and $c = 1$.

   Solution: $2 \cdot 4 \cdot 3 \cdot 3 + 3 \cdot 4 \cdot 4 \cdot 3 + 4 \cdot 1 = 72 + 144 + 4$
   $= 220.$

   Note that the numbers are substituted for, or put in place of, the letters.

Using the values of a, b, and c given in Example 1, evaluate each of the following expressions:

2. $a^2 + b^2 + c^2$
3. $3abc$
4. $a^2b + ab^2$
5. $ac^2 + cb^2 + ba^2$
6. $a^3 + b^3 + c^3$
7. $\frac{a}{b} + \frac{b}{a} + \frac{c}{b}$
8. $\frac{a - b + c}{a + b - c}$
9. a^2bc^2
10. $\frac{4}{a} + \frac{3}{b} + \frac{1}{c}$
11. $\frac{a^3 - b^3 - c^3}{2a^2}$
12. $a^b + b^a + c^a$
13. $a^2 + bc + ac^2$
14. $abc^2 + ab^2c + a^2bc$
15. $a^2b^2c^2$
16. $b^2 + 2bc + c^2$
17. $\frac{a}{b} + \frac{b}{c} + \frac{c}{b}$
18. $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}$
19. $2ab + 2a^2c + 2ac^2$
20. The formula $d = 16t^2$ tells what distance an object will fall in any number of seconds. Find how far a body will fall in 1 sec. of time, that is, when $t = 1$. Do you believe it? How could you test it?
21. Using the formula in Example 20, find how far an object will fall in 2 sec. of time, that is, when $t = 2$. How could you test the truth of this?

22. The horsepower of an automobile is given by the following formula: $\frac{D^2N}{2.5}$, in which D represents the diameter of the piston, and N the number of cylinders. What is the horsepower of a Ford, which has 4 cylinders, and in which $D = 3\frac{1}{2}$ in.?

III. THE CONSTRUCTION OF FORMULAS

Section 19. It is very important to be able to make a formula for any computation that must be performed over and over again. For example, we often have to find the area of a square. Instead of saying or writing each time

> **"the area of a square is equal to the square of the number of units in one of its sides,"**

it saves time to use the formula $A = s^2$, in which $A =$ area and $s =$ one of the sides. **This formula tells all that the word rule says and requires much less effort.** To give practice in this kind of work, construct a formula for each of the examples in the following exercise.

EXERCISE 22

1. (*a*) Find the volume of a rectangular box whose dimensions are 12, 8, and 6 inches. What did you do to the 12, 8, and 6 to find the volume? If l, w, and h stand for the dimensions of any box, what would you do to them to find the volume?

(*b*) Write the formula for the volume of any box.

2. (*a*) What is the area of a circle whose radius is 9 in.?

(*b*) What did you have to do to the 9 to get the area? If the radius is represented by R, what would you have to do to get the area? Write a formula for the area of any circle.

3. (*a*) Draw, or imagine, a cube each edge of which is 8 inches. Find the entire surface of all the faces of the cube.
   (*b*) What did you do in (*a*) to find the entire surface? Just what would you do if each edge were s units long?
   Write the formula for the entire surface of any cube.

4. (*a*) What is the interest on \$400 for 2 years, at 6 %?
   (*b*) Just how did you solve (*a*)? How would you find the interest on p dollars for t years at r %?
   Write a formula for the interest on any principal for any rate and for any time.

5. (*a*) How many cubic inches in a block 2′ by 3′ by 4′?
   (*b*) Make a formula for the number of cubic inches in any rectangular solid whose dimensions are expressed in feet.

6. Make a formula for, or an equation which tells, the cost of any number of pounds of beans at 12 cents per pound.

7. What equation or formula will represent the area of any rectangle whose base is 5 inches, but whose height is unknown? Evaluate your formula for $h = 3.4$.

8. An automobilist travels 20 miles per hour. How far does he go in 2 hours? in 5 hours? What do you do to the number of hours to get the distance? If time is t instead of 5, what must you do to the t to get the distance? What formula or equation will represent the distance he travels in t hours? Evaluate this formula for

$$t = 5 \text{ hr. } 20 \text{ min.}$$

9. You learned in arithmetic how to find the product of two fractions; thus $\frac{2}{5} \times \frac{3}{7} = \frac{2 \cdot 3}{5 \cdot 7}$. What is the rule for finding the product of any two fractions? Express this rule as a formula, using $\frac{a}{b}$ and $\frac{c}{d}$ for any two fractions.

10. In arithmetic you learned how to divide one fraction by another: *e.g.* $\frac{3}{4} \div \frac{5}{7} = \frac{3}{4} \times \frac{7}{5} = \frac{3 \cdot 7}{4 \cdot 5}$. Using $\frac{a}{b}$ and $\frac{c}{d}$ for the fractions, state the rule for division of fractions as a formula.

11. Solve each of the following fractional equations:

(*a*) $\frac{5}{7}x = 35$

(*b*) $\frac{4}{5}n = 12$

(*c*) $\frac{2}{3}y = 120$

(*d*) $\dfrac{x}{60} = \dfrac{1}{4}$

(*e*) $\frac{2}{23}y = 4$

(*f*) $\dfrac{120}{5y} = 6$

(*g*) $\dfrac{90}{4x} = 2.5$

12. Make a formula for each of A, b, and h, in which A is the area of a triangle, and b and h are the base and height, respectively.

13. Using the formula $i = prt$, find what principal will yield \$60 interest in 3 years at 4%.

14. Determine the value of V in the formula $V = lwh$, if $l = 5.4$, $w = 4.8$ and $h = \frac{1}{3}w$.

HOW TO USE THE TIMED PRACTICE EXERCISES

On all **timed exercises** start all pupils working at the same moment; stop them **exactly** at the end of the allowed time. (Set your watch exactly at some even hour, as at 9 o'clock and 0 minutes.)

The teacher will read the correct answers and the pupils should correct their own work. Pupils should then record, on the record sheet on page 49, or on their individual card, the number of examples attempted and the number right, and compare their rights with the "**standard.**" The authors secured the best results by giving a given "timed" practice exercise every third or fourth day until pupils reached the standard; after that, once in two weeks will be found sufficient to hold the skill.

Pupils must not write answers on the text. For reasons obvious to the teacher it is desirable to vary the example with which to begin on a given trial. Occasionally it will help to begin with the last example and work back toward the first.

It is very important to keep a record of one's practice by trials. Progress in learning is graphed in Chapter VII.

PRACTICE EXERCISE A (TIMED) *

The examples in this exercise have been worked by pupils in many high schools. On the average they

* These and subsequent Practice Exercises are from the authors' *Standardized Practice Exercises*, which are now used in 300 cities. They

succeeded in doing 10 examples right in 4 minutes on the fourth or fifth trial. Can you do this on your fifth trial?

1. If $c = 4$ and $f = 2$ what does $2c^2 - 3cf$ equal?
2. If $a = 3$ and $b = 2$ what does $3ab + ab^2$ equal?
3. If $x = 3$ and $y = 4$ what does $xy^2 - 2xy$ equal?
4. If $c = 5$ and $d = 2$ what does $\frac{c^2}{5} + \frac{4cd}{4}$ equal?
5. If $r = 2$ and $s = 4$ what does $r^2 + 3r^2s$ equal?
6. If $d = 3$ and $e = 4$ what does $4d^2 - 2de$ equal?
7. If $m = 2$ and $n = 3$ what does $2mn + mn^2$ equal?
8. If $a = 4$ and $b = 5$ what does $ab^2 - 4ab$ equal?
9. If $x = 4$ and $y = 3$ what does $\frac{x^2}{2} + \frac{4xy}{6}$ equal?
10. If $p = 3$ and $q = 5$ what does $p^2 + 2p^2q$ equal?
11. If $a = 3$ and $b = 2$ what does $3a^2 - 2ab$ equal?
12. If $r = 4$ and $s = 2$ what does $2rs + rs^2$ equal?
13. If $u = 3$ and $v = 4$ what does $uv^2 - 3uv$ equal?
14. If $x = 5$ and $y = 2$ what does $\frac{x^2}{5} + \frac{3xy}{6}$ equal?
15. If $b = 3$ and $c = 2$ what does $b^2 + 3b^2c$ equal?

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STANDARDIZED PRACTICE EXERCISES in FIRST YEAR ALGEBRA

PRACTICE RECORD CARD :: Rugg and Clark

Name: ______________________ *Age:* ______ *School:* ______________

EXER-CISE	No. of trial:			No. of trial:			No. of trial:			No. of trial:			No. of trial:		
	Date	Att's.	R'ts.	Date	Att's.	R'ts.	Date	Att's.	Rt's.	Date	Att's.	Rt's.	Date	Att's.	R'ts.
A															
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FIG. 12

EXERCISE 23

PRACTICE IN MAKING ALGEBRAIC EXPRESSIONS

1. A and B together have \$30. A has n dollars; how many has B?

2. A man can do a certain piece of work in 10 days. What part can he do in 1 day? in 4 days? in x days? in $2y$ days?

3. John's age is now 30 years. What expression will represent his age in 5 years? in x years from now? n years ago?

4. A newsboy received 40 cents for n papers. What did he receive for each? for 10?

5. The difference between two numbers is d. If n is the smaller one, what is the larger? What expression will stand for the sum of the two numbers?

6. A rectangular field is x rods long and y rods wide. What is the area of the field in square rods? in acres? What will stand for the value of the field if it is worth n dollars per acre?

7. A lot is x feet wide and 60 feet long. What will stand for its perimeter? its area?

8. How long will it take a man to ride n miles if he rides 12 miles an hour?

9. Express algebraically the sum of a and b; the difference between a and b; the product of a and b; and the quotient of a and b.

10. James sold a motorcycle for \$175, thereby gaining x dollars. What did it cost him?

11. Two men, A and B, are 30 miles apart. They start to walk toward each other, A at the rate of 3 miles per hour and B at the rate of 2 miles per hour. How fast do they approach each other? In how many hours will they meet?

12. If, in Example 11, A and B had been x miles apart, in how many hours would they have met?

13. What are consecutive numbers? If 10 is the smaller of two consecutive numbers, what represents the larger?

14. If n is the middle of three consecutive numbers, what represents the other two numbers?

REVIEW EXERCISE 24

1. What does an equation express? Is $7 + 4 = 6 + 6$ an equation?

2. Does $2n + 1 = 21$, if $n = 9$, make an equation? if $n = 10$?

3. The formula for the perimeter of a rectangle, $p = 2a + 2b$, contains three *unknown* numbers. How many of them must be *known* in order to use this formula to solve an example?

4. Read each of the following equations as questions, and find the value of the unknown number:

(*a*) $4y + 3 = 21$ (*d*) $3y - 5 = 16$
(*b*) $20 = 6 + 2x$ (*e*) $x + x = 36$
(*c*) $5c + 2 = 42$ (*f*) $b + b + 1 = 23$

5. Three times a certain number, plus 2, equals 38. Find the number.

6. Donald saved twice as much money as his older brother. Express in algebraic language that both together saved \$ 96. How much did each save?
7. Evaluate the formula $V = l^3$ ($V =$ the volume of a cube), if $l = 4\frac{1}{2}$.
8. The first of three numbers is twice the second, and the third is twice the first. Find each number if their *sum* is 105.
9. Construct a formula for the cost of any number of eggs at 30 cents per dozen.
10. What is the difference in meaning between $10\,n$ and $n + 10$? Does $4\,w$ mean the same as $4 + w$?

SUMMARY OF CHAPTER III

The most important principles and methods which we have learned in this chapter are the following:

1. **A formula is merely a shorthand rule of computation.**
2. **Algebraic expressions are "evaluated" or "solved" by substituting numbers for the letters in the expression.**
3. **We have frequent need to be skillful in substituting numbers for letters in practical formulas.**
4. **Exponents are used as short methods of indicating multiplication. An exponent of a number tells how many times that number is taken as a factor.**
5. **We should construct a formula for any kind of problem which we have to solve frequently.**

REVIEW EXERCISE 25

1. Make a formula for the number of revolutions made by the front wheel of a Ford car in going a mile, if the radius of the wheel is 14 inches.

2. In what sense does the equation $7b-5=b+25$ ask a question?

3. Give one illustration of the advantage of using letters for quantities.

4. What are the four fundamental principles or axioms which are used in solving equations?

5. Does $x=4$ satisfy the equation $x^2-3x=6$?

6. Using m, s, and d for minuend, subtrahend, and difference, respectively, what equation or equations can you make from them?

7. An autoist travels at an average rate of 24 mi. per hour. What distance will he cover in 2 hr.? in 5 hr.? in 10 hr.? Make an equation or formula for the distance he will travel in t hr.

8. Write a formula for the cost of any number of pounds of bacon at 45 cents per pound.

9. Draw rectangles with bases of 2 inches each, but with different heights. What is done to the height to get the area? Then, if h represents the height, what must be done to it to get the area?

   What formula will represent the area of any such rectangle if h represents the height?

10. How do you get rid of fractions in an equation? What is the most convenient multiplier in any particular equation?

CHAPTER IV

HOW TO FIND UNKNOWN DISTANCES BY MEANS OF SCALE DRAWINGS: THE FIRST METHOD

Section 20. We need to know how to find unknown distances. The methods of mathematics are really all planned to help us find *unknown* values. The equation, which we have studied so carefully, is the best *algebraic* tool with which to do that. **Many times,** however, in practical life work the **unknown values** that we need to know **are distances.** For example, the surveyor may need to know the distance across a river and may not be able actually to measure it. Or, he may need to know the distance between two points, with some other intervening object between which prevents him from measuring it directly. Now, mathematics has given us **three ways to find such an unknown distance.** In Chapters IV, V, and VI we shall discuss these methods.

The **first method** is to **make a scale drawing,** which will include in some way the unknown distance. Next, therefore, we shall study *how to determine unknown distances by means of scale drawings.* Before we take up that particular subject, however, we must study *how to measure the lines and angles* which make up scale drawings.

Section 21. The measurement of lines. We are already familiar with certain methods of measuring distances. For example, we have measured the length of lines, such as the distance from A to B or from C to D. If we use a metric *scale,* in which the units are *centimeters,* the distance from

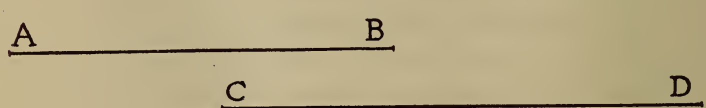


FIG. 13

A to B, which is read "*line* AB," is 5.08 centimeters long, and the line CD is 6.35 centimeters long. If we use a foot rule in which the units are inches, the distance between A and B, or the line AB, is 2 inches, and line CD is 2.5 inches. Note here that the distances or lengths that we obtain for these lines depend upon the kind of *scale*, or kind of *unit*, that is used in measuring.

THE MEASUREMENT OF ANGLES

Section 22. An angle is determined by one line turning about a point in another line. In order to construct scale drawings, we must know how to measure angles. Let us think of an angle as being formed by one line turning, or rotating, about a fixed point on some fixed or stationary line. The line OY turns or rotates about point O. For example,

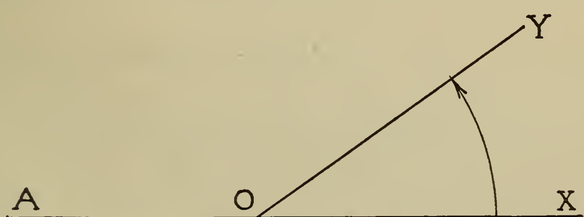


FIG. 14

in Fig. 14, think of AX as a fixed, or stationary, line. (It is easiest always to take this line as *horizontal.*) Think also of another line, say OY, as turning, or rotating, about some point on the fixed line AX, say point O. As the line OY rotates about the point O, it constantly forms a larger and larger angle with the fixed line AX. (**The symbol for angle is $\angle$.**) The point O, about which the line turns, is always the point at which the two sides of the angle meet, and is called the **vertex of the angle.**

The arrow is drawn to indicate that **the line *OY* is turning, or rotating, about the point *O*.**

Section 23. The unit of angular measurement. Just as we have **units** and **scales** for measuring straight lines, so **we have units and scales for measuring angles.** Evidently the **unit** with which we must measure the size of the angle is one that will measure **the amount that the line has rotated about the fixed point.** Figure 15 shows that we can

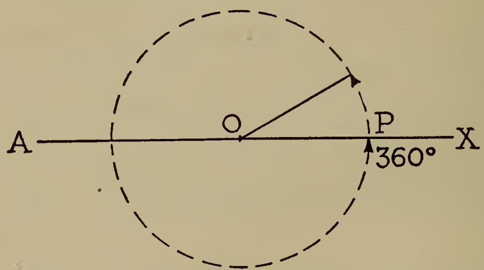


FIG. 15

think of the rotating line as turning clear around until it occupies its original position again. That is, any point P on the line OX has turned through a complete *circle* in rotating about O and returning to its original position.

This suggests that the **unit** with which we measure **angles will be some definite fraction of the circle.** For a long time people have agreed that the circle be divided into **360 units** and that each one of these *units of angular measure* be called a DEGREE. The symbol used for degree is a small ° placed at the right above the number. For example: 45° is read "45 degrees." Thus, Figs. 16, 17, and 18 illustrate angles of different sizes or of different numbers of degrees.

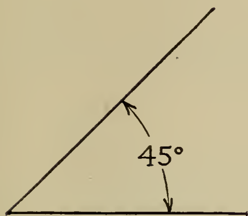


Fig. 16

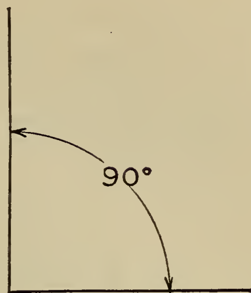


Fig. 17

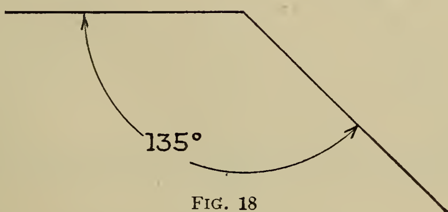


Fig. 18

Section 24. The PROTRACTOR: How to measure angles. Just as we use foot rules, yardsticks, meter sticks, etc., to measure straight-line distances, so we have an instrument called a PROTRACTOR to measure angular distances. Figure 19 shows that the circular edge of the

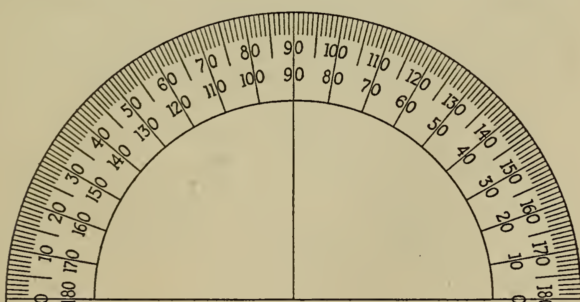


Fig. 19. **A protractor for constructing and measuring angles.**

protractor is marked off (*i.e.* is "graduated") into degrees. Note from the figure that the protractor is divided into 180 equal parts (half of the total number of angular units in the circle), called **degrees.** Sometimes the whole circle is used and marked off to give 360°.

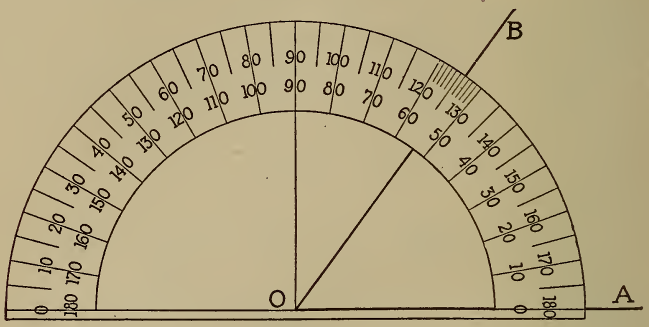


FIG. 20

The next figure, Fig. 20, shows **how to measure an angle** with a protractor. First, lay the straight edge of the protractor so that it will fall exactly upon one of the two lines that form the angle, and with the center of the protractor exactly upon the VERTEX, *O*, of the angle. Then the other side of the angle, *OB*, for example, will appear to cut across the circular edge of the protractor. Now count the number of degrees from the point where the curved edge of the protractor touches *OA* to the point where it crosses the line *OB*. Hence, in Fig. 20, the angle *AOB* contains 54°. It is very important for us to be able *to read angles accurately.* The next exercise will give you practice in reading angles.

EXERCISE 26

PRACTICE IN MEASURING ANGLES

1. Measure each of these angles with a protractor in the way described in the last paragraph.

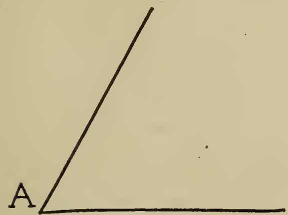


FIG. 21

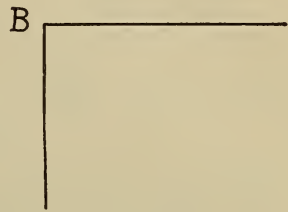


FIG. 22

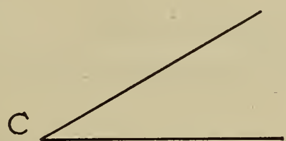


FIG. 23

2. Compare angle *A* and angle *C*. Which has the longer sides? What effect has the length of a side of an angle upon the size of the angle?

3. Measure each angle of triangle *ABC*. From the results of your measurement, **what is the sum of all three angles of this triangle?**

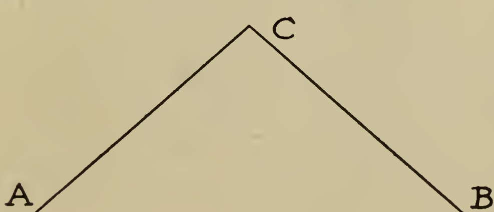


FIG. 24

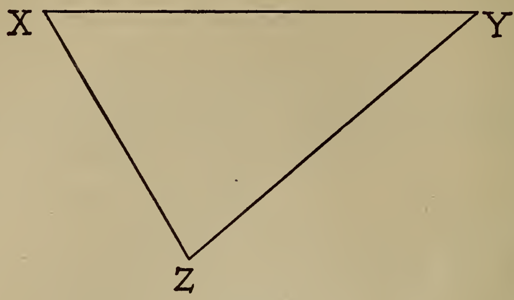


FIG. 25

4. How large is $\angle x$? How many degrees in $\angle y$? in $\angle z$? How many degrees in the **sum** of the angles of **this** triangle, XYZ?

5. Draw with the protractor an angle of 30°; 45°; 60°; 100°.

6. At each end of a line 3 in. long draw angles of 50°. Produce these lines until they meet, and measure the angle formed by them. How many degrees in it? Compare the lengths of the lines you drew. How many degrees does the **sum** of the three angles of **this** triangle make?

7. Draw a triangle such as triangle ABC, so that $AB = 4$ inches, angle $A = 60°$ and $AC = 3$ inches. Then find the number of degrees in angle B and angle C.

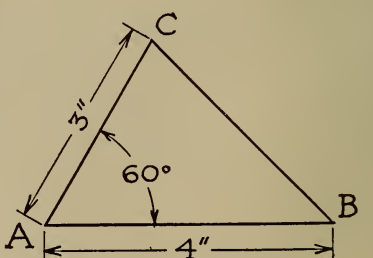


FIG. 26

8. Construct triangle ABC so that $AC = 4$ in., angle $C = 40°$, and $CB = 4$ in. Compare angle A with angle B. How many degrees in each?

Explanation: A triangle having two sides equal, such as AC and CB, is an *isosceles triangle*. It is proved in geometry that the angles opposite these equal sides are always equal; for example, angle A = angle B. How many degrees ought there to be in either angle A or angle B?

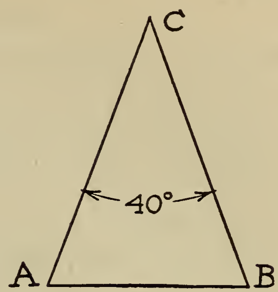


FIG. 27

Section 25. How to describe an angle. An angle is *described* by using three letters, *i.e.* the letter which represents the vertex is written between the two letters at

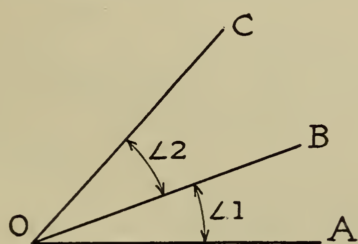


FIG. 28

the ends of the sides. Thus, $\angle 1$, in Fig. 28, is read as *angle AOB* or *angle BOA*, and is written $\angle AOB$ or $\angle BOA$. In the same way, $\angle 2$ is read *angle BOC* or *angle COB*, and is written $\angle BOC$ or $\angle COB$.

EXERCISE 27

PRACTICE IN READING ANGLES

1. Why would it not be clear to read ∠ 2 as ∠ *O*?
2. Read the angle formed by lines *OA* and *OB*.
3. Read the angle formed by lines *OB* and *OC*.
4. Determine the number of degrees in ∠ *AOB*, in Fig. 29, **without using the protractor.**

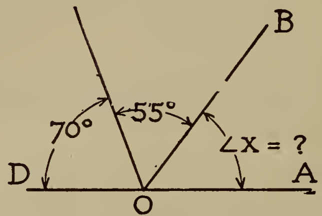


FIG. 29

5. If in Fig. 30 you know that angle *ABC* is 40° and that ∠ *BCA* is 90°, could you find ∠ *CAB* without measuring it? How? How large is it?

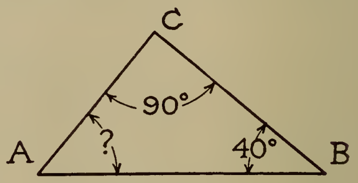


FIG. 30

Section 26. The sum of the angles of any triangle is 180°. You have measured the angles of several triangles and found that the sum is very close to 180°. If your measurements had been absolutely correct, you would have obtained exactly 180° for the sum of the angles of each triangle. In geometry it is proved without measurement, that the sum is always 180°. You will use this fact in solving the examples in the next exercise.

EXERCISE 28

PRACTICE IN FINDING THE VALUES OF ANGLES IN A TRIANGLE

1. The number of degrees in each of the three angles of a triangle is represented by x, $x+10$, and $2x$, respectively. Find the size of each angle.

2. In triangle ABC, $\angle C$ is equal to $\angle B$, and $\angle A$ is equal to the sum of $\angle B$ and $\angle C$. Find the number of degrees in each angle.

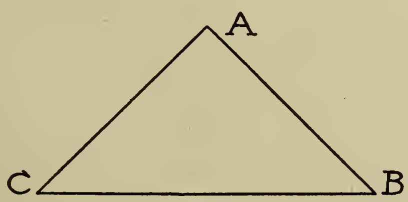


FIG. 31

3. The first angle of a triangle contains x degrees; the second angle is 15° larger than the first, and the third is as large as the sum of the other two. Find each angle.

4. Angle A in a certain triangle is twice as large as $\angle B$, and $\angle C$ is 10° larger than the sum of $\angle A$ and $\angle B$. How large is each angle?

5. One angle of an isosceles triangle is 50°. Find the size of the angles opposite the equal sides. See Example 8, page 61.

6. Find the angles of a triangle if the first is one half of the second, and the third is three fourths of the second.

7. Find the angles of a triangle in which the second angle is $10°$ larger than the first, and the third is $10°$ larger than the sum of the first and second.

8. Angle B is $10°$ less than $\angle A$; $\angle C$ is twice as large as $\angle B$. How large is each angle?

Section 27. We must be able to find unknown distances which cannot be measured directly. The preceding section took up only examples in which the distances, linear and angular, could be measured directly, by means of instruments. There are many instances, however, in which the lengths of the lines and the sizes of the angles cannot be measured directly. For example, consider the case of finding the distance across a river, or the height of a tree, which we mentioned at the beginning of the chapter. In cases like this we need **indirect methods of measuring.** Mathematics makes it possible for us to determine the lengths of such lines **by measuring the lengths of other lines and the sizes of angles that are related to them.** This leads us to the **main topic of this chapter.**

HOW TO FIND UNKNOWN DISTANCES BY MEANS OF SCALE DRAWINGS

Section 28. How to draw distances to scale. One of the methods that you will use commonly in indirect *measurement* is that of drawing distances "to scale." So much use is made of mechanical drawings that we need to be very proficient in making them and in reading them. Let us take a simple illustration of the drawing of distances "to scale" and of measuring distances on scale drawings.

Illustrative example. A man starts at a given point and walks 2.5 miles east, then 2.5 miles north. How far is he from his starting point?

First, set point *O*, in Fig. 32, as his starting point. *East* is measured to the *right* of point *O* and *north above* point *O*.

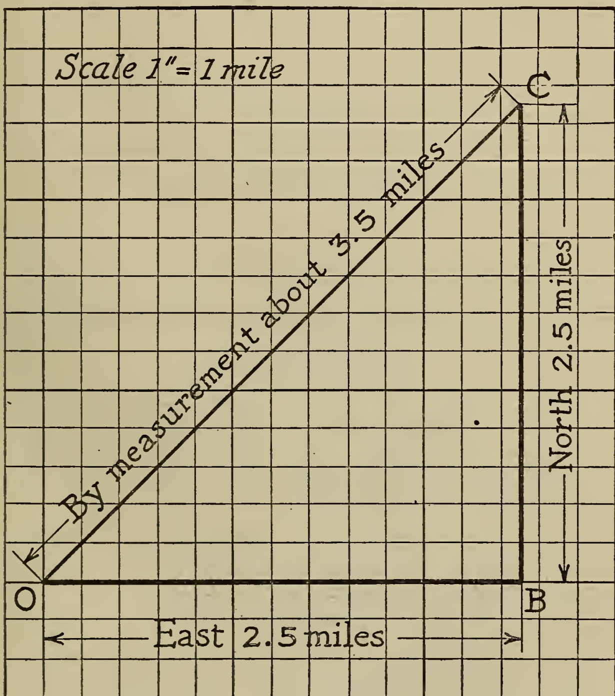


FIG. 32

Second, to represent distances "to scale," we need to *select a unit of distance on the scale which will represent a unit of distance in the example*. Let us take, for example, an inch on the drawing to represent each mile which the man actually walks. This is indicated on the scale drawing (Fig. 32) by writing "Scale = 1 in. to 1 mi." It is very important to select the

scale unit carefully and always to indicate the scale that has been used on the drawing.

Third, to represent the man's path, we lay off OB horizontally to the right of O, 2.5 miles (on the drawing this amounts to $2\frac{1}{2}$ inches) and BC vertically, 2.5 miles. Then, by using the cross-section paper as a scale, *we can measure at once the distance*, OC, that the man is from his starting point. The distance is 3.54 inches on the figure, or 3.54 miles.

This work illustrates by a very simple example how we make **scale drawings**. Mechanical drawings made "to scale" are used very commonly by such workers as architects, carpenters, machinists, and engineers.

EXERCISE 29

PRACTICE IN FINDING UNKNOWN DISTANCES BY THE CONSTRUCTION OF SCALE DRAWINGS

1. Draw to the scale 1 cm. to 2 ft. a floor plan of a room 28 ft. by 20 ft. By measuring the distance diagonally across the plan, compute the diagonal of the room.

2. Draw a plan of a baseball diamond 90 ft. square and find the distance from first base to third base. Use 1 cm. to represent 20 ft.

3. Two bicyclists start from the same point. One rides 12 miles north and then 8 miles east; the other rides 10 miles south and then 6 miles west. How far apart will they be? Use the scale 1 cm. to 2 mi.

4. Draw to the scale 1 cm. to 4 ft. a plan of the end of a garage such as in Fig. 33. Find the height from the floor to the top of the roof.

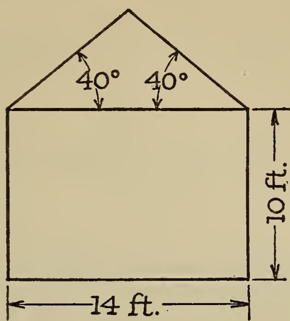


FIG. 33

5. A surveyor sometimes finds it necessary to measure the distance across a swamp, such as *AB* in Fig. 34. He measures from a stake

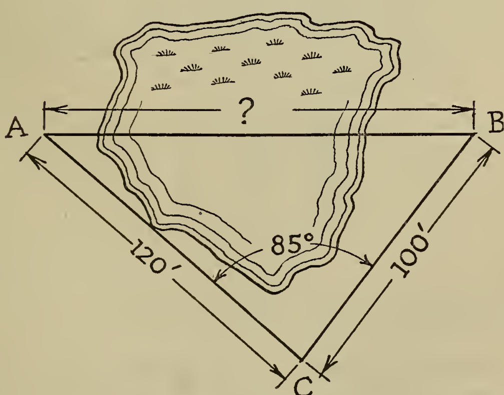


FIG. 34

A to a stake *C*, 120 ft. From *C* to *B* he finds it is 100 ft. Find, by a scale drawing, the distance *AB* across the swamp, if angle *C* is 85°.

6. How could a surveyor find the distance from A to B, if there were some obstacle in the way

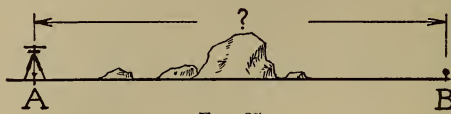


FIG. 35

preventing his measuring directly the distance AB? Could he represent AB as a line in a triangle from which he could make a scale drawing?

7. Find by a scale drawing the distance AC across

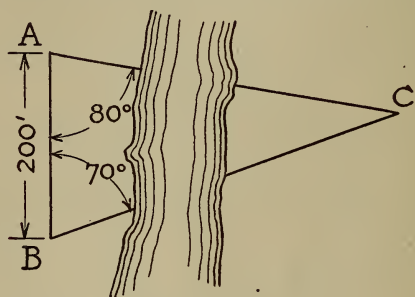


FIG. 36

the river, if it is known that angle $A = 80°$, $AB = 200$ ft., and angle $B = 70°$.

8. **Illustrative example. A boy wishes to determine the height of a flagpole. A scale drawing will aid him in doing this. For example, he can measure a line of any length on the ground out from the base of the flagpole. Suppose he takes a line 80 feet long. Then he sets at *E* an instrument called a *transit*, with which he can read the angle between the horizontal base line and the *line of sight* from *E*, where he stands, to *H*, the top of the pole. He**

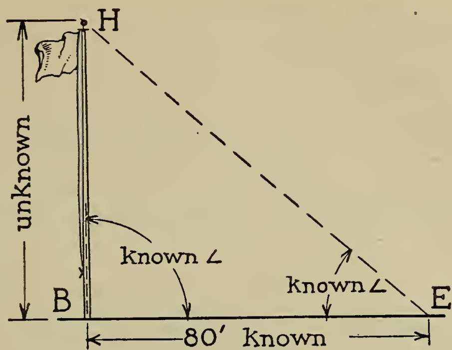


FIG. 37

knows also that the angle *B* is a right angle. So he knows the length of the line *EB*, the size of the angle *E* and the angle *B*. He constructs a scale drawing to represent the known length and the known angles. From this drawing he is able to "scale" or measure the height of the flagpole.

The angle *BEH* or angle *E* between the horizontal and the line of sight in this example is called the **ANGLE OF ELEVATION.** If the boy had taken a longer base line, what would have been true of the size of the angle of elevation with respect to what it was before?

9. If the angle of elevation in Fig. 37 is 40° when the observer is 80 ft. from the foot of the pole, what would be the height of the pole?

10. A flagpole 50 ft. high casts a shadow 60 ft. long on level ground. What is the angle of elevation of the sun? If the length of a shadow cast by this pole increases, what conclusion can be drawn concerning the angle of elevation?

11. The angle of elevation of the top of a tree is 42° when the observer stands 30 yd. from the tree. How high is the tree? If the distance from the observer to the tree decreases, what change in the angle of elevation follows?

12. An anchored balloonist from a height *HT*, Fig. 38, of 2500 yd., observes the enemy at *D*.

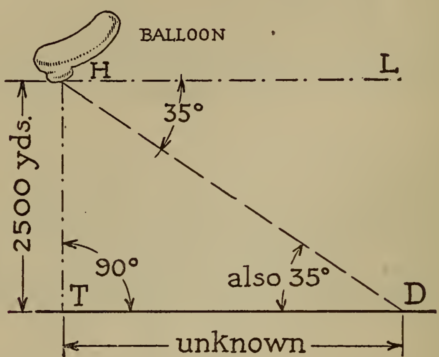


Fig. 38

He wishes to compute the distance *DT*, on level ground. To do so, he measures the angle which is formed by the horizontal line *HL* and the line of sight *HD*. This angle is called the ANGLE OF DEPRESSION, and has the same number of degrees as the angle of elevation. (Can you see that this is true?) He next finds angle *THD* by subtracting the angle of depression from 90°, angle *THL*. He then knows enough about the triangle *DTH* to make a scale drawing of it. Find *DT* if the angle of depression is 35°.

13. From the top of a lighthouse 80 ft. high the angle of depression of a ship is 35°. How far is the ship from the base of the lighthouse? Compare your result with that obtained by the other members of the class.

14. From the top of a cliff 120 ft. above the surface of the water the angle of depression of a boat is 20°. How far is it from the top of the cliff to the boat?

15. An observer is 200 ft. from the ground. The angle of depression of a point *A* is 24°, of a point *B* 42°, and of a point *C* 15°. Which point is closest to the observer? farthest from the observer?

16. In Fig. 39 measure (1) the angle of elevation of point *D* from point *E*; (2) the angle of depression of point *E* from point *D*. Compare these angles. Note that line *DH* is parallel to *EC*, and that these angles are formed by line *DE* cutting (or intersecting) these two parallels. In geometry it is proved that such angles are always equal.

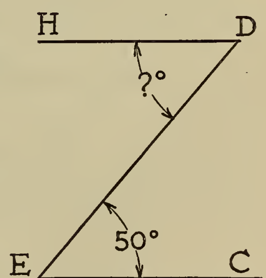


FIG. 39

17. Is the scale drawing a very *accurate* method of determining unknown distances? Do many of the pupils in the class get the same answer for any particular example? Why?

SUMMARY OF CHAPTER IV

The most important things you have learned in this chapter are the following:

1. **It is possible to find unknown distances by making a "scale drawing" of the known distances and angles involved in the problem, and measuring or "scaling" the unknown on the drawing.**
2. **To do this we need to know how to measure angles as well as lines. Much practice has been given in the use of (1) the angular unit, *i.e.* the degree; and (2) the instrument for measuring angles: the protractor.**

REVIEW EXERCISE 30

1. A boy knows that AB is 100 ft. and that $\angle B = 40°$. From this information can he construct a scale drawing for the triangle? Give reasons for your answer.

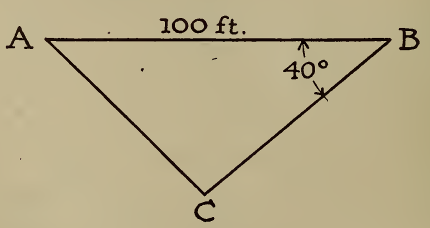


FIG. 40

2. What facts or data must be known about a triangle before you can make a scale drawing of it?
3. A tree 90 ft. high casts a shadow 140 ft. long. Find from a scale drawing the angle of elevation of the sun.

4. Give the meaning of each of the following terms: exponent, factor, multiple, equation, scale, angle, and formula.

5. Represent in the briefest way the product of five x's; the sum of five x's.

6. Does $a^2b = ab^2$ if a is 4 and b is 3?

7. Can you construct a scale drawing for the triangle in Fig. 40 if you know that AB is 200 ft., $\angle B = 40°$, and $AC = 150$ ft.? Why?

8. What must be known about a triangle before you can construct it accurately?

9. Find x in $6x - 2 = 19$.

10. Solve for y: $16 + 5y = 50$.

11. A man earned p dollars per month, and spent s dollars per week. Make a formula for his yearly savings.

12. Evaluate the formula in example 11 when $p =$ \$150 and $s =$ \$25.

13. Make a more complete summary of this chapter than is given above.

14. If a square is made 6 inches wider, and 5 inches longer, the resulting figure will then have a perimeter of 122 inches. How much more area will it then have?

15. A train is traveling x miles per hour. If it increases its speed 5 miles per hour, it will then travel 240 miles in 6 hours. Find its original rate.

16. One number is 8 more than another. Two thirds of the smaller, plus the larger, equals 28. Find each number.

17. Angle A in a certain triangle is two fifths of angle B. Find each angle, if the third angle is $54°$.

18. In an isosceles triangle, the base is two thirds of each of the equal sides; its perimeter is 80. Find the base.

19. A certain angle is bisected. One of the angles obtained lacks 30 degrees of being half of a right angle. How large was the angle which was bisected?

CHAPTER V

A SECOND METHOD OF FINDING UNKNOWN DISTANCES: THE USE OF SIMILAR TRIANGLES

Section 29. Scale drawings are somewhat inaccurate. Hence it is better to use similar triangles. In the last chapter we saw that scale drawings could be used to find unknown distances, either *linear* or *angular*. The **results obtained, however, were very inaccurate.** Seldom did many of you get the same answer for any example. Therefore **we need more accurate methods** *for determining unknown lines and unknown angles.* This chapter, and the next one, will show methods that depend less upon the accuracy or skill of the person who "scales" or measures. **The first method,** *based upon geometrical figures of exactly the same shape*, will be explained now.

Section 30. What are similar figures? You have already seen many objects or figures of exactly the same shape. A scale drawing has the same shape as the figure from which it was made; on a photographic plate the figure is the same in outline or shape as the original; the map of a state has the same shape or outline as the state itself.

Figures which have the same shape are said to be SIMILAR FIGURES. Which of the following figures are similar in shape?

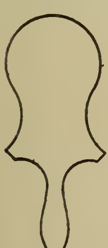


FIG. 41

FIG. 43

FIG. 44

FIG. 45

FIG. 42

FIG. 46

Similarity in shape in geometrical figures is a very important principle that we are able to use in many ways in mathematics. Before this can be taken up, however, you need to know more about **RATIO** than you learned in arithmetic. This important method of comparing quantities will now be taken up.

A NEW WAY TO COMPARE TWO QUANTITIES; NAMELY, TO FIND THEIR **RATIO**

Section 31. We have been comparing two quantities by finding how much larger or smaller one quantity was than another quantity. For example, if the line AB is 4 units long and the line CD is 6 units long, we should have said, in describing the comparative lengths, that the line AB was 2 units shorter than line CD, or that line CD was 2 units longer than the line AB. Here the unit is $\frac{1}{2}$ inch.

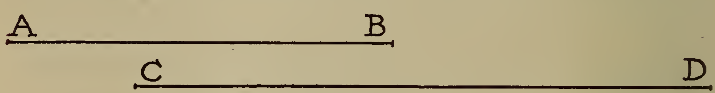


FIG. 47

Another method, however, of comparing quantities is used very extensively in mathematics. It is the method of dividing one quantity by the other, or finding the quotient of the two quantities. Thus, to compare the line AB with the line CD we divide AB by CD, which gives:

$$\frac{AB}{CD}=\frac{4}{6}.$$

This result is read "the quotient of AB and CD, or the RATIO of AB to CD is equal to four sixths." This process is described as finding the RATIO of the two lines. The **ratio** of two numbers means, then, the quotient which results from dividing one of the numbers by the other. Thus, the ratio of 5 to 10 is $\frac{1}{2}$, and is written $\frac{5}{10}=\frac{1}{2}$. The ratio of 10 to 5

is 2, and is written $\frac{10}{5} = 2$. In the same way the ratio of 1 in. to 1 ft. is $\frac{1}{12}$; the ratio of $\frac{1}{2}$ to $\frac{3}{4}$ is $\frac{2}{3}$; and the ratio of $4x$ to $7x$ is $\frac{4x}{7x}$ or $\frac{4}{7}$.

EXERCISE 31

PRACTICE IN DEALING WITH RATIOS OF NUMBERS

1. What is the ratio (in lowest terms) of 10 to 12? of 20 to 24? of 15 to 18? of 25 to 30? Show that $5x$ and $6x$ represent all pairs of numbers whose ratio is $\frac{5}{6}$.
2. Give several pairs of numbers having the ratio $\frac{3}{4}$. Show that $3x$ and $4x$ represent all pairs of numbers having this ratio.
3. The ratio of two numbers, a and b, is $\frac{5}{4}$. What is b when a is 40?
4. The ratio of two lines, m and n, is $\frac{3}{2}$. Find the length of m if n is 18 in.

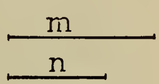


FIG. 48

5. Divide 40 into two parts whose ratio is $\frac{6}{4}$.
6. A father and his son agreed to divide the profits from their garden in the ratio of $\frac{7}{3}$. Find each one's share if the total profits were $210.
7. The ratio of $\angle A$ to $\angle B$ is $\frac{5}{3}$. Find $\angle B$ when $\angle A$ is 80°.

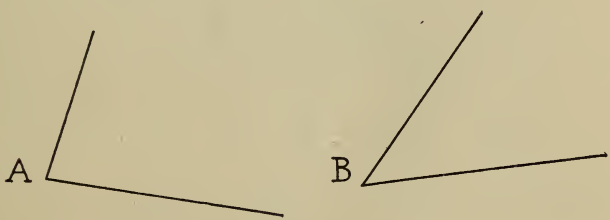


FIG. 49

8. The ratio of the areas of the two squares, S_1 and S_2, is $\frac{4}{1}$. Find the area of each if the sum of their areas is 45 sq. in.

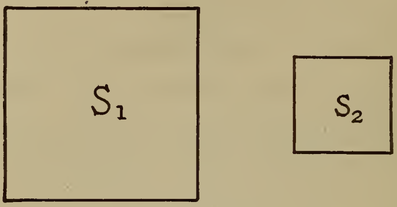


FIG. 50

9. Divide an angle of 90° into two angles having the ratio of 4 to 5.

10. Measure each angle, Fig. 51, with a protractor. Find their ratio.

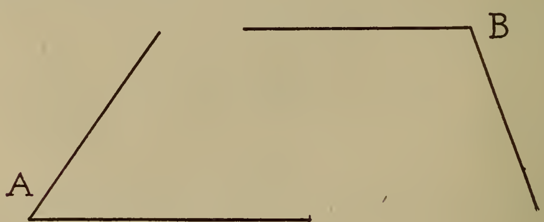


FIG. 51

11. In the two triangles ABC and XYZ, what is the ratio of $\angle C$ to $\angle Z$? of $\angle A$ to $\angle X$? of $\angle B$

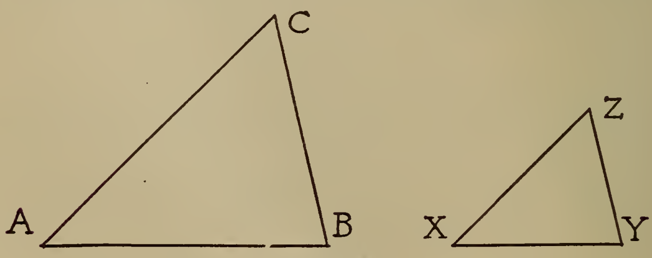


FIG. 52

to $\angle Y$? Do these triangles have the **same shape**? Do all triangles have the **same shape**?

12. Find the ratio of two lines if one is 2 feet long and the other 3 yards long.

13. What is the ratio of 3 pints to 4 quarts?

14. $\frac{2}{n} = \frac{5}{15}$. **15.** $\frac{4}{3} = \frac{n}{6}$. **16.** $\frac{5}{7} = \frac{10}{n}$.

17. Find the ratio of AB to CD, Fig. 53, by measuring the length of each line. Express the result decimally.

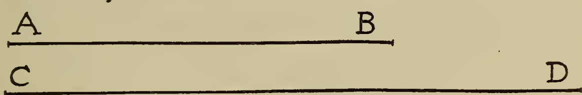


FIG. 53

18. A school baseball team won 7 of the 10 games it played. What was its standing in percentage?

Solution: The standing of the team in percentage means the ratio of the number of games won to the number played, *expressed decimally.* Thus, the standing of this team is expressed by the result obtained from changing the ratio, $\frac{7}{10}$, to a decimal. This gives .70 or .700.

19. What was the percentage or standing of a team which won 9 of its 12 games?

20. The winning team in one league won 14 of its 18 games; and the winning team in another league won 15 of the 19 games it played. Which league had the higher percentage?

21. Express each of the following ratios as a decimal, correct to two places:

(*a*) $\frac{3}{5}$ (*c*) $\frac{5}{8}$ (*e*) $\frac{5}{16}$ (*g*) $\frac{21}{28}$ (*i*) $\frac{22}{81}$

(*b*) $\frac{2}{7}$ (*d*) $\frac{7}{11}$ (*f*) $\frac{4}{9}$ (*h*) $\frac{32}{45}$ (*j*) $\frac{16.2}{21}$

22. If 12 quarts of water are added to 25 gallons of alcohol, what is the ratio of the water to the entire mixture? Express decimally.

Section 32. Similar triangles. The next exercise is intended to show a very important fact about triangles which have the same shape, *i.e.* about similar triangles.

EXERCISE 32

PRACTICE WITH SIMILAR TRIANGLES

1. Draw a line XY twice as long as AB, Fig. 54. At X draw an angle equal to angle A. At Y draw an angle equal to angle B. Produce the sides of these angles until they meet at Z. Measure the angle formed by these sides. How should it compare with angle C? Why?

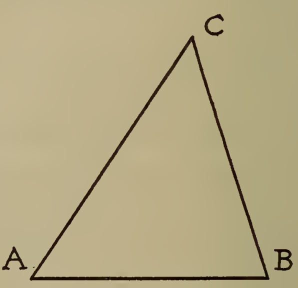


FIG. 54

2. (*a*) Angle A *corresponds* to what angle in your triangle?

(*b*) Angle B *corresponds* to what angle in your triangle?

(*c*) What is true, then, about the *corresponding angles* of the two triangles?

3. Measure the side in your triangle which *corresponds* to AC, and the side which *corresponds* to BC. What is the ratio of AB to XY, or what is the value of $\frac{AB}{XY}$? of $\frac{AC}{XZ}$? of $\frac{BC}{YZ}$? What does this tell about **the ratios of corresponding sides**?

4. Construct a triangle larger than Fig. 55 but having its angles equal to the angles of Fig. 55. Is your triangle the same shape as Fig. 55?

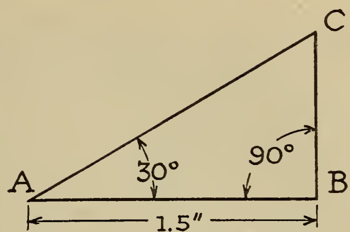


FIG. 55

After careful measurement find the ratio of AB to its **corresponding side** in your triangle. Then find the ratio of AC to its **corresponding side**. Compare these ratios.

IMPORTANT PRINCIPLE

The previous experiments in finding the ratios of the sides of triangles of the same shape show the following important principle:

> **If it is known that the angles of one triangle are equal respectively to the angles of another triangle, then it follows that the ratios of the corresponding sides are equal.**

This principle or truth is used very much in mathematics. To illustrate, suppose we know that the angles of one triangle are equal **respectively** to the angles of another triangle; then we **also know** that the **ratios of corresponding sides are equal.** Hence, we can make an **equation** from these equal ratios and from this equation find important unknown distances. The next exercise will show how this is applied to finding the length of lines.

A proof, which is not based on measurement, is commonly given for this fact, in the second year's work in mathematics.

EXERCISE 33

ADDITIONAL PRACTICE WITH SIMILAR FIGURES

1. Figure 56 is a right triangle. Why? If angle A is 30°, find angle C.

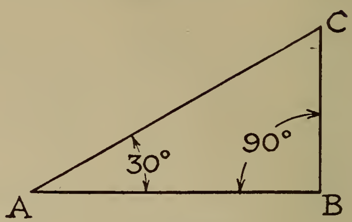


FIG. 56

2. In a right triangle one of the acute angles (that is, one of the angles smaller than a right angle) is 40°. Find the other acute angle.

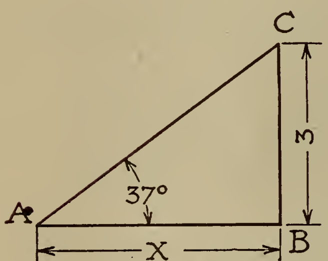


FIG. 57

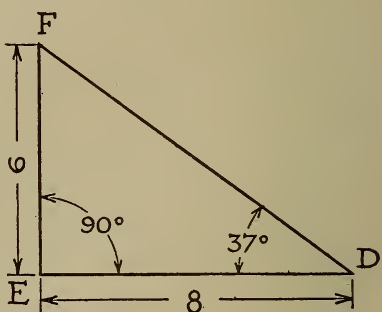


FIG. 58

3. Figures 57 and 58 are right triangles. If angle $A =$ angle D, are the triangles similar? Why?
4. In Figs. 57 and 58, if $EF = 6$, $ED = 8$, and $CB = 3$, what must AB equal? To solve this problem we use the principle that the ratios of the corresponding sides of similar triangles are equal. This gives the equation $\frac{x}{8} = \frac{3}{6}$. What, therefore, is AB?

5. The sides of a triangular plat of ground are 150 ft., 100 ft., and 125 ft., respectively. The side of a scale drawing of this plat, corresponding to the 150-foot side, is 5 cm. Find the side of the scale drawing corresponding to the 100-foot side. Solve as in Example 4.

6. The sides of a triangle are 3, 4, and 5 cm. The shortest side of a similar triangle is 16 cm. Find the other sides of the second triangle.

7. A house is 36 ft. high and the garage is 16 ft. high. If the house is represented in a drawing as 18 in. high, how high should the drawing of the garage be? What mathematical principle is used to show this?

8. Two rectangular gardens are the same shape, but of different size. The larger one is 72 ft. by 84 ft. If the length of the smaller one is 40 ft., what must be its width?

9. Two angles of one triangle are equal respectively to two angles of another triangle. Are the triangles similar? Why?

10. Line AB is **parallel** to line CD. Would they meet if produced, either to the right or to the left of the third line MN? Measure $\angle 1$ and $\angle 2$. These angles are called **corresponding angles** of parallel lines.

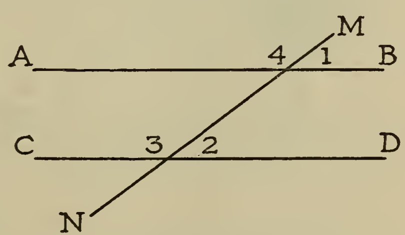


FIG. 59

11. In Fig. 59 measure another pair of **corresponding angles,** $\angle 3$ and $\angle 4$. What do you find?

These two exercises illustrate a very important fact in mathematics; namely, that **the corresponding angles of parallel lines are always equal.** Later on this will be proved without measuring the angles; that is, without any possibility of error. You will make use of this fact without again measuring the angles.

12. In triangle ABC, DE is drawn parallel to AB. Does $\angle 1 = \angle 2$? Why? Is triangle DEC **similar** to triangle ABC? Why?

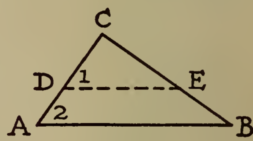


FIG. 60

13. In Example 12, $DC = 12$, $AC = 21$, and $CE = 14$. Show how BC can be found, by using the principle that the ratios of corresponding sides of similar triangles are equal. What is the length of BC?

14. A boy wishes to measure the height of a tree. He notes that the tree, AC, its shadow, AB,

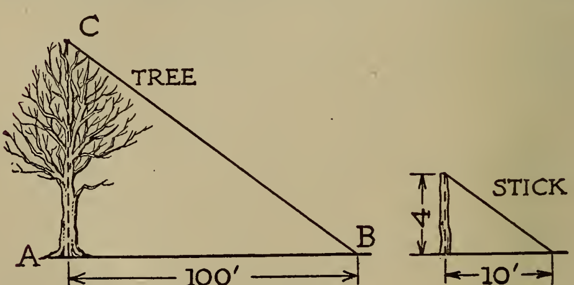


FIG. 61

and the sun's ray, CB, passing over the top of the tree, form a triangle. He measures the shadow and finds it 100 ft. long. At the same

time a vertical stick 4 ft. high makes a shadow 10 ft. long. Why is the triangle formed by the stick, its shadow, and the sun's ray passing over the top of the stick similar to the other triangle? How can the boy find the height of the tree from the similar triangles? What is its height?

15. A Boy Scout wagered he could find the distance between two trees, *A* and *B*, on opposite sides of a river, without crossing it. Could he do it, and if so, how? If not, why not?

16. A crude way to measure the height of an object is by means of a mirror. Place a mirror horizontally on the ground at *M*, and stand at the point at which the image of the top of the

FIG. 62

object is just visible in the mirror. Show how, by measuring certain distances, this would enable one to compute the height of the object.

17. In triangle *ABC*, *DE* is parallel to *CB*. Show that triangle *AED* is similar to triangle *ABC*. If $BC = 10$, $ED = 5$, and $AE = 8$, what is AB? Draw the figure.

18. Figure 63 shows two triangles, with the size of each angle indicated, which a teacher drew

upon the blackboard for an examination. She asked the following questions about the two triangles:

(*a*) Are they similar triangles? Why?

(*b*) Does $\frac{AB}{ZY}=\frac{AC}{XY}$? Why?

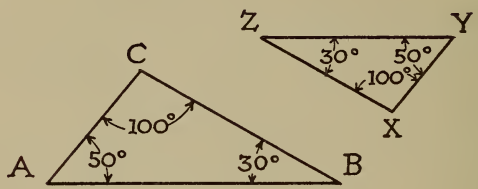


FIG. 63

(*c*) Does $\frac{AC}{XY}=\frac{BC}{ZY}$? Why?

(*d*) Does $\frac{ZY}{AB}=\frac{XZ}{BC}$? Why?

(*e*) Does the ratio of *any* two sides equal the ratio of *any other* two sides?

How would you have answered these questions?

19. The sides of a small triangle are 3, 4, and 6. Do you think it is similar to a larger triangle whose sides are 15, 18, and 30?

SUMMARY OF THE IMPORTANT POINTS OF CHAPTER V

It is important to have clearly in mind the following important conclusions from the chapter:

1. **If you know that the angles of one triangle are equal respectively to the angles of another triangle, then you know that *the ratios of the corre-***

sponding sides are equal. In other words, you can make an equation, and thereby find an unknown side.

2. **The corresponding angles of parallel lines are equal.**
3. **Unknown distances may be found by means of similar triangles.**

REVIEW EXERCISE 34

1. Translate into words: $4y + 3 = y + 21$.
2. If A, B, and C represent the number of degrees in the respective angles of a triangle, we know that $A + B + C = 180°$. Why? What is A if $B = 40°$ and $C = 65°$?
3. If five times a certain number is divided by 2.7, the result is 3. What is the number?
4. Given the formula $V = lwh$, find a formula for l; for w.
5. A boy receives C cents an hour for regular work, and pay for time and a half when he works overtime. What will represent his earnings for 6 hr. overtime? Evaluate this when $C = 50$.
6. If n represents a boy's present age, what is the meaning of the expression $n - 5 = 11$? of $n + 4 = 20$?
7. The sides of one triangle are 10, 12, and 15 inches; the sides of a similar triangle are 8, x, and 12 inches. Determine the length of x, the side which corresponds to the 12-inch side in the first triangle.

8. One number is three times as large as another; two fifths of the large number added to one half of the smaller number gives 17. Find each number.

9. The second angle of a triangle is 15° less than the first angle; the third angle is one sixth of the first. Find each angle.

10. One number is 5 larger than another; their ratio is $\frac{3}{2}$. Find each number.

11. Room A in a certain school gave twice as much to the Red Cross as Room B; Room C gave \$4 more than half as much as Room B. What was given by each room, if all three gave \$39?

12. Does $x^2 + 5x = 14$ make an equation if $x = 2$? if $x = 3$? if $x = 4$?

CHAPTER VI

HOW TO FIND UNKNOWNS BY MEANS OF THE RIGHT TRIANGLE

Section 33. The advantage of the RIGHT TRIANGLE in finding unknowns. It should be clear by this time that mathematics gives us methods of finding unknown quantities. The equation is the most important tool for doing this, for the reason that when we solve a problem we have to make an equation. This equation must contain the unknown quantity together with other known quantities which are related to it in some way.

In the last chapter we saw that an equation could be formed from the *ratios of corresponding sides of similar triangles* and that by that means we could find an unknown length. Two facts, however, make that method less satisfactory than the one we shall study in this chapter: (1) we must always be certain the triangles are *similar*, or we have no right to make an equation, and (2) the method is cumbersome because we must always use *two* triangles.

There is a **particular kind of triangle** whose properties can be used to find unknown distances **accurately** and at the same time more easily than by any other method. It is the **RIGHT TRIANGLE.** The most important facts about the right triangle are found in connection with first, the *ratios of its different sides;* second, the relation between the hypotenuse and the other two sides. We shall first discuss the ratios of the different sides.

A. FINDING UNKNOWNS BY USING THE RATIOS OF THE SIDES

I. THE TANGENT OF AN ANGLE

Section 34. The ratio of the "side opposite" a given angle to the "side adjacent" the given angle, *i.e.* the

TANGENT of the angle. You will recall that in a right triangle one angle is 90° and the sum of the two **acute angles** equals 90°. (Why?) In finding unknown distances by means of right triangles we shall always deal especially with one of the **acute** angles. Therefore, in referring to the sides of a right triangle, when dealing with a given angle, we shall speak of them as they are described in Figs. 64 and 65. If angle B is the acute angle with which

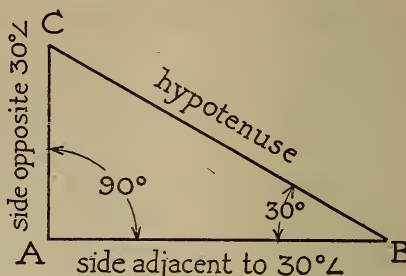


FIG. 64

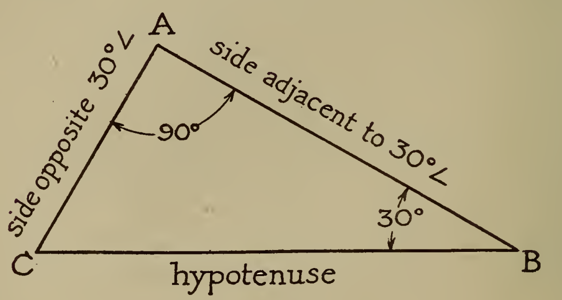


FIG. 65

we are concerned, then side AC is the "side opposite" $\angle B$, and side AB is the "side adjacent" $\angle B$. The side opposite the 90° angle is **always** called the **HYPOTENUSE.**

Some exercises will show the importance of the **ratio** of

$$\frac{\text{the "side opposite"}}{\text{the "side adjacent"}}$$

an acute angle of a **right triangle.**

EXERCISE 35

SOME MEASUREMENTS TO FIND THE NUMERICAL VALUE OF THE RATIO OF THE "SIDE OPPOSITE" TO THE "SIDE ADJACENT" A 30° ANGLE OF A RIGHT TRIANGLE

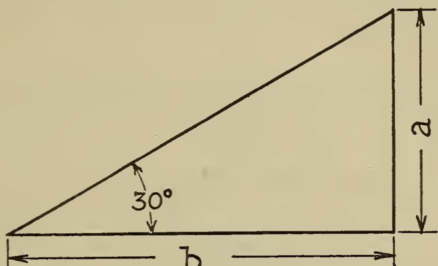


FIG. 66

1. In Fig. 66, a is the "side opposite" the 30° angle and b is the "side adjacent" the 30° angle. Measure a and b. Now find the numerical value of the ratio of a to b by dividing the length of a by the length of b. Record your results in Table 1.

2. In Fig. 67, a and b are respectively the "side opposite" and the "side adjacent" an acute angle of 30°. Measure each and compute the ratio $\frac{a}{b}$ to two decimal places. Record your results in Table 1.

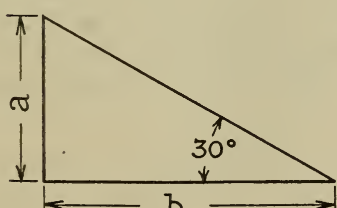


FIG. 67

3. Draw any other triangle similar to those above, but with much larger sides. Measure the "side opposite" and the "side adjacent" the 30° angle and compute their ratio as before. Record results, as before, in Table 1.

   TABLE 1. Record here the results of measuring the sides of right triangles and of computing the ratio of the "side opposite" to the "side adjacent" an **acute angle of 30°.**

TABLE 1

	Length of a	Length of b	Ratio of a to b (i.e., $\frac{a}{b}$)
Fig.			
Fig.			
Fig.			
Fig.			

   What do you notice in the table about the numerical values of the ratios

$$\frac{\text{the "side opposite" an acute angle of } 30^\circ}{\text{the "side adjacent" an acute angle of } 30^\circ} \text{ or of } \frac{a}{b}?$$

   The members of the class should compare results, to see what result seems most likely to be the true one. If great care is taken in measuring, the ratio should be very close to **.58 in each triangle.** Why should it be **the same** in each triangle?

4. In Fig. 68, *CB*, *DE*, and *GF* are perpendicular to *AB*. Is triangle *AFG* similar to triangle *AED*? Why? Is either of the smaller triangles sim-

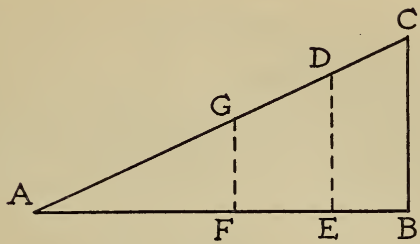


FIG. 68

ilar to the large triangle? Why? From this, why does the ratio of GF to AF, or $\frac{GF}{AF}$, equal the ratio of DE to AE, or $\frac{DE}{AE}$? If you measured these lines, and computed the ratios, what would you expect to be true of the results?

Section 35. This last example is very important, because it shows, without measurement, that the ratio $\frac{GF}{AF}$ equals the ratio $\frac{DE}{AE}$. But this is the same as saying that **the ratio of the "side opposite" to the "side adjacent" a 30° angle in one right triangle is ALWAYS equal to the ratio of the "side opposite" to the "side adjacent" a 30° angle in any other right triangle.** The length of the sides may be far different, but the **ratios** should be the same. This shows that the ratios obtained in the table **should** have been the same, if it were possible to draw and measure without error.

We shall now make use of the fact that the numerical value of the ratio of the "side opposite" to the "side adjacent" an acute angle of 30° (in a right triangle) is approximately .58.

EXERCISE 36

1. Illustrative example.

A man wishes to determine the height of a smokestack. He finds that the angle of elevation of the top of the smokestack, from a point 200 ft. from the base of the smokestack, is 30°.

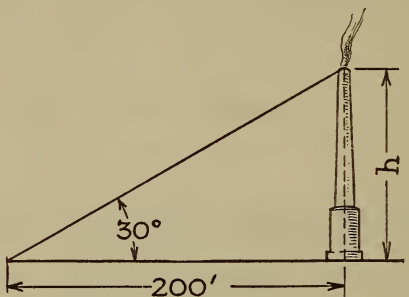


FIG. 69

Solution: $\frac{h}{200} = .58.$ **(Why?)**

$h = 200 \times .58.$ **(Why?)**

$h = 116$ **ft.**

Note here that $\frac{h}{200}$ is the ratio of the "side opposite" to the "side adjacent" the 30° angle. From previous work we know that this ratio is .58. Thus, we can make the equation $\frac{h}{200} = .58$.

2. In triangle ABC, angle A is 30° and angle C is 60°. Find CB if AB is 70 yards. What is CB if AB is 10 inches? Draw the figure.

3. In triangle XYZ, angle X is 30° and angle Z is 60°. Find XY if YZ is 116 ft.

4. In triangle DEF, angle E is 30° and angle D is 90°. Find ED if DF is 24 in.

5. In Fig. 70, CD bisects angle C and is perpendicular to AB. How many degrees in angle BCD? If CD is 100 cm., how long is DB?

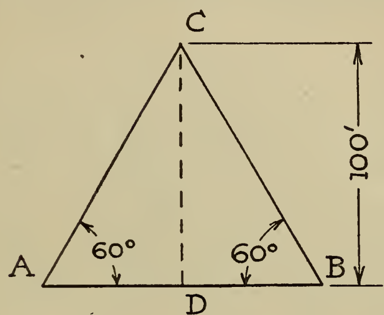


FIG. 70

6. In the right triangle ABC, angle B is 60°, angle A is 30°, and BC is 50 ft. Find AC.

7. In triangle XYZ, angle X is 30°. Does the ratio $\dfrac{ZY}{XY} = .58$? State definitely when this ratio is equal to .58.

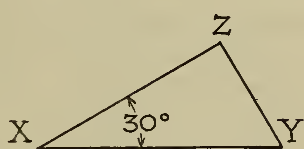


FIG. 71

Section 36. It is convenient to name important ratios. Since it is helpful to use the ratios of the various sides of a **right triangle**, very frequently in finding unknown distances, each is given a definite name. The ratio of the "side opposite" an acute angle to the "side adjacent" is called:

THE TANGENT OF THE ANGLE

Its abbreviation is **tan.** Thus, in the above examples the **tangent of an angle of 30° is constant; it is approximately .58.**

EXERCISE 37

1. Construct a right triangle such as Fig. 72, with AB equal to 4 cm. and angle A equal to 40°. Then measure BC and from that find the tangent of an angle of 40°. Compare results with those of other members of the class.

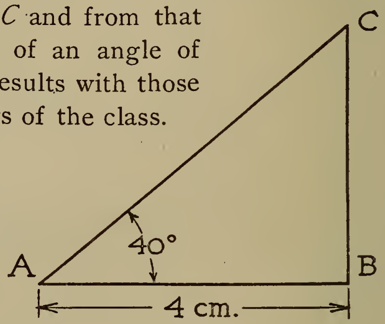


FIG. 72

2. In a similar way find the tangent of an angle of 50°. (Use AB as 4 cm.) Also find the tangent of each of the following angles: 60°, 70°, and 20°.

Section 37. Summary of steps in finding the tangent of an angle. These examples show how to find the tangent of any angle. **Three steps are necessary**; namely: **(1)** measure the side opposite the particular angle; **(2)** measure the side adjacent the angle; **(3)** divide the first number obtained by the second. To do this, however, for angles of all sizes from very small to very large, would require a great deal of labor, and probably give, for a great many of you, inaccurate results. To save this trouble, and at the same time get very accurate results, these ratios or tangents have been computed very carefully and compiled in a table like Table 4. (See Table of Tangents on page 97.)

TABLE 2

TABLE OF COSINES AND TANGENTS

NUMERICAL VALUES OF THE TANGENTS AND COSINES OF THE ANGLES FROM 0° TO 90° INCLUSIVE

Deg.	tan	cos	Deg.	tan	cos
0	.000	1.000	46	1.04	.695
1	.017	.999	47	1.07	.682
2	.035	.999	48	1.11	.669
3	.052	.999	49	1.15	.656
4	.070	.998	50	1.19	.643
5	.087	.996			
			51	1.23	.629
6	.105	.995	52	1.28	.616
7	.123	.993	53	1.33	.602
8	.141	.990	54	1.38	.588
9	.158	.988	55	1.43	.574
10	.176	.985			
			56	1.48	.559
11	.194	.982	57	1.54	.545
12	.213	.978	58	1.60	.530
13	.231	.974	59	1.66	.515
14	.249	.970	60	1.73	.500
15	.268	.966			
			61	1.80	.485
16	.287	.961	62	1.88	.469
17	.306	.956	63	1.96	.454
18	.325	.951	64	2.05	.438
19	.344	.946	65	2.14	.423
20	.364	.940			
			66	2.25	.407
21	.384	.934	67	2.36	.391
22	.404	.927	68	2.48	.375
23	.424	.921	69	2.61	.358
24	.445	.914	70	2.75	.342
25	.466	.906			
			71	2.90	.326
26	.488	.899	72	3.08	.309
27	.510	.891	73	3.27	.292
28	.532	.883	74	3.49	.276
29	.554	.875	75	3.73	.259
30	.577	.866			
			76	4.01	.242
31	.601	.857	77	4.33	.225
32	.625	.848	78	4.70	.208
33	.649	.839	79	5.14	.191
34	.675	.829	80	5.67	.174
35	.700	.819			
			81	6.31	.156
36	.727	.809	82	7.12	.139
37	.754	.799	83	8.14	.122
38	.781	.788	84	9.51	.105
39	.810	.777	85	11.4	.087
40	.839	.766			
			86	14.3	.070
41	.869	.755	87	19.1	.052
42	.900	.743	88	28.6	.035
43	.933	.731	89	57.3	.017
44	.966	.719	90	Inf.	.000
45	1.000	.707			

EXERCISE 38

FINDING ANGLES AND TANGENTS FROM THE TABLE OF TANGENTS

Find, from Table 4, each of the following:

1. tan 42°.
2. The angle whose tangent is .58.
3. tan 57°.
4. The angle whose tangent is .94.
5. tan 14°.
6. The angle whose tangent is $\frac{5}{4}$.
7. tan 25°.
8. The angle whose tangent is $\frac{2}{3}$.
9. tan 45°.

EXERCISE 39

EXAMPLES WHICH INVOLVE THE USE OF THE TANGENT OF AN ANGLE

1. **Illustrative example.** **The brace wire AC of a telephone pole BC, Fig. 73, makes with the ground an angle of 62°. It enters the ground 15 ft. from the foot of the pole. Find the height of the pole BC.**

Solution:

$\frac{BC}{AB} = \text{tangent } 62°.$

$\frac{BC}{15} = 1.88$ (from the Table).

$BC = 15 \times 1.88 = 28.2.$

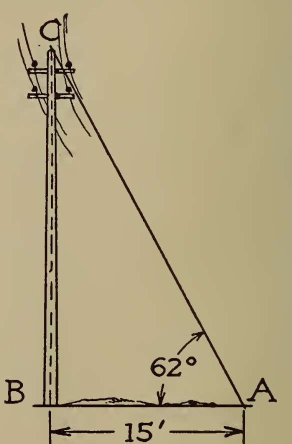


FIG. 73

2. The angle of elevation of the top of a tree, from a point 75 ft. from its base (on level ground), is 48°. How high is the tree?

3. From a vertical height of 1500 yd. a balloonist notes that the angle of depression of the enemy trench is 51°. Find the distance from the trench to the point on the level ground directly below the balloonist. Make a drawing.

4. The angle of elevation of an aëroplane at point A on level ground is 44°. The point B on the ground directly beneath the aëroplane is 450 yd. from A. How high is the aëroplane?

5. If a flagpole 42 ft. high casts a shadow 63 ft. long, what is the angle of elevation of the sun?

6. In Fig. 74, CD is perpendicular to AB. Find AD if angle $A=60°$ and $CD=20$.

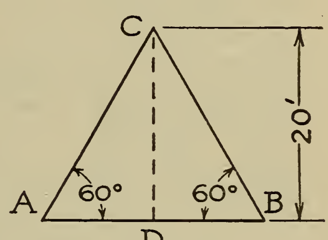


FIG. 74

7. From the point of observation on a merchant vessel, the angle of depression of the periscope of a submarine was 17°. How far was the submarine from the merchant vessel, if the observer was 40 ft. above the water?

8. Turn back to page 71 and solve problem 13 by this method. How do your results compare with those obtained by scale drawings?

9. Solve Example 12, page 70, by this method. Which method is preferable?

Section 38. There are many problems which cannot be solved by means of the tangent. In the previous section we found that the **ratio** of the "side opposite" to the

"side adjacent" an acute angle of a right triangle **is always constant** for any particular angle. This enabled us to find the length of the sides and the size of the acute angle. Now we come to another fact about right triangles. Let us examine a problem which *cannot* be solved by the use of the tangent.

Illustrative problem. **In Fig. 75, *BC* represents a telephone pole, *AC* an anchor wire, and *AB* the distance from the foot of the pole to the point at which the wire enters the ground, 20 ft. The wire makes an angle of 30° with the ground. How long is the wire?**

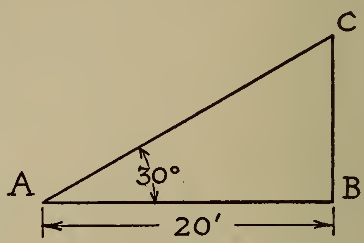


FIG. 75

II. THE COSINE OF AN ANGLE

Section 39. The ratio of the "side adjacent" the given angle to the hypotenuse of the triangle, *i.e.* the COSINE.

Clearly, AC, in the above example, cannot be found by means of the ratio which we called the tangent, because the tangent of 30° makes use only of BC and AB, and we must get a ratio which contains AC. Therefore, to solve this problem we shall have to **learn how to use the ratio of the "side adjacent" the 30° angle, to the hypotenuse, or $\frac{AB}{AC}$.**

EXERCISE 40

EXPERIMENTS TO DETERMINE THE NUMERICAL VALUE OF THE RATIO $\frac{\text{THE "SIDE ADJACENT" A } 30^\circ \text{ ANGLE}}{\text{THE HYPOTENUSE OF THE TRIANGLE}}$, *I.E.* THE *COSINE*

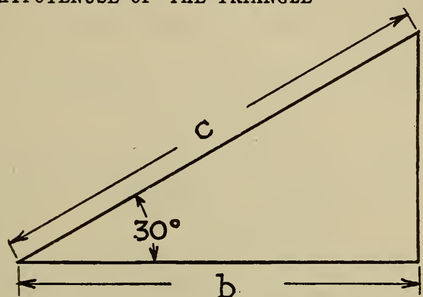


FIG. 76

1. Measure the length of b and c in Fig. 76. Then compute the ratio $\frac{b}{c}$ to two decimals.
2. Draw any other triangle similar to Fig. 76, but with much longer sides. Find, as in Example 1, the ratio of the side adjacent the 30° angle, to the hypotenuse. Compare your result with that of Example 1.

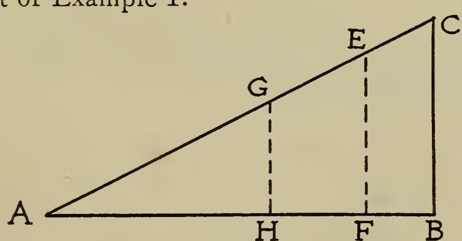


FIG. 77

3. In Fig. 77, EF and GH are perpendicular to AB. Why does $\frac{AH}{AG} = \frac{AF}{AE} = \frac{AB}{AC}$?

Section 40. The COSINE of a particular angle is CONSTANT. This shows that the ratio of the "side adjacent" a 30° angle to the hypotenuse of one right triangle is equal to the same ratio in any other right triangle **which has an acute angle of 30°.** For this reason, you would get the same numerical value for $\frac{b}{c}$ in Examples 1 and 2, if it were not for errors in measurement.

Therefore, just as in the case of the tangent, so the cosine, *i.e.* $\frac{\text{“side adjacent” } 30^\circ \text{ angle}}{\text{hypotenuse}}$, is always constant, when the angle is 30°. It is approximately .86. *The right triangles may differ in size and position, but as long as they are similar* (that is, so long as the acute angles we are dealing with are the same size), *this ratio does not change.*

EXERCISE 41

PROBLEMS SOLVED BY APPLYING THE CONCLUSION ARRIVED AT ABOVE; NAMELY, THE RATIO OF THE "SIDE ADJACENT" A 30° ANGLE TO THE HYPOTENUSE IS .86.

1. **Illustrative example.** The anchor wire AC, of a telephone pole, meets the ground 20 ft. from the foot of the pole, making an angle of 30° with the ground. Find the length of the wire AC.

Solution: $\frac{AB}{AC} = .86$. (Why?)

$\frac{20}{h} = .86$.

$20 = .86\,h$, or $h = 23.2$ ft.

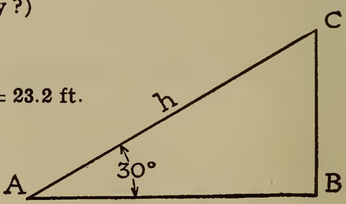


FIG. 78

2. The rope, AC, of the flagpole, BC, makes an angle of 30° with the ground, at a point 42 ft. from the foot of the pole. How long is the rope? Make a drawing.

3. The angle of elevation of the top of a tree from a point A, on level ground, 100 ft. from the base of the tree, is 30°. What is the distance from A to the top of the tree?

4. In the right triangle ABC, AB is 25 in. and $\angle A = 30°$. Find AC.

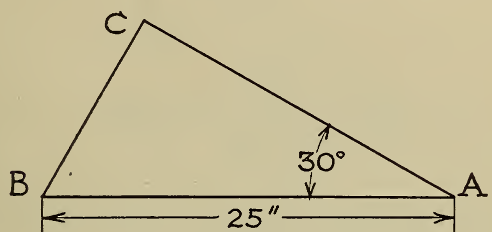


FIG. 79

5. Draw a right triangle such that angle $A = 60°$ and the hypotenuse $AC = 4$ in. From this could you find BC?

6. The angle of depression of a boat, from the top of a cliff, is 30°. Find the distance from the observer to the boat, if the boat is 400 ft. from the foot of the cliff.

These examples have been solved by using the ratio of the "side adjacent" an acute angle of 30° to the hypotenuse, or, as we shall call it from now on, by using the **COSINE OF THE ANGLE.** The abbreviation for cosine is **cos**. Thus,

$$\cos \angle B = \text{ratio of } \frac{\text{"side adjacent" } \angle B}{\text{hypotenuse of the triangle}}.$$

EXERCISE 42

1. Construct a right triangle similar to Fig. 80, with $A = 40°$ and $AB = 4$ cm. Then measure c and compute the ratio $\frac{b}{c}$. By comparing your result with cos 40° as given in the table, see if you are within .05 of the correct result.

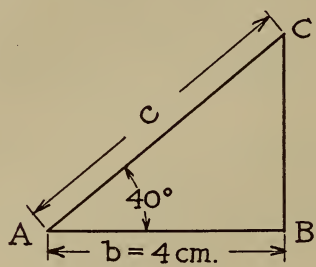


FIG. 80

2. How would you construct or draw the cos of a 60° angle? of an 80° angle?

3. Read from the table of cosines:
   (*a*) cos 67°.
   (*b*) The angle whose cos is .258.
   (*c*) cos 45°.
   (*d*) The angle whose cos is .573.
   (*e*) cos 2°.
   (*f*) The angle whose cos is .707.
   (*g*) cos 89°.
   (*h*) The angle whose cos is .629.

4. A surveyor desires to measure the distance BC across a swamp. He surveys the line BA perpendicular to BC. He extends this line BA until he can measure from A to C. If AC is 400 ft.

and angle C is 55°, show how he would compute the length of BC. Find BC.

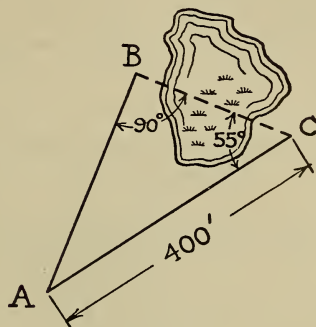


FIG. 81

5. A boy observes that his kite has taken all the string, 750 ft. Assuming that the string is straight and that it makes an angle of 34° with the ground, how far on level ground is it from the boy to the point directly below the kite?

6. The angle of elevation of the top of a tent pole, from a point 43.2 ft. from the foot of the pole, is 32°. Find the distance from the point of observation to the top of the pole.

7. Figure 82 is a right triangle. Find AB if angle A is 42° and $AC = 67$. HINT: What is the cosine of angle A?

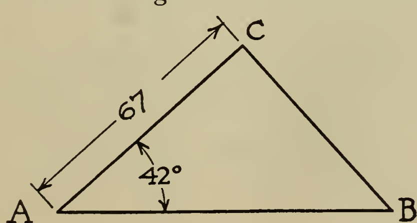


FIG. 82

8. How long a rope will be required to reach from the top of a flagpole to a point 19 ft. from the foot of the pole (on level ground) if the rope makes an angle of 63° with the ground?

9. The angle of depression of a boat from the top of a cliff is 37° when the boat is 1260 ft. from the foot of the cliff. Find the distance from the boat to the top of the cliff.

10. Find angle A if AB is 27 and AC is 48. HINT: What is $\frac{27}{48}$ with respect to angle A?

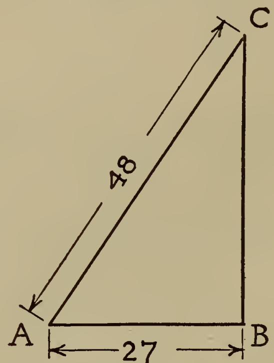


FIG. 83

11. A man starts at O and travels in a direction which is 48° east of a north-south line. How far due north of O will he be when he is 26 miles from O?

12. From the table find the cosine of 32°. Then find the cosine of an angle twice as large as 32°, and see if it is twice as large as the cosine of 32°. Does the cosine of an angle change or vary in the same way that the angle changes or varies?

EXAMPLES IN WHICH NEITHER THE COSINE NOR THE TANGENT OF THE GIVEN ANGLE MAKES USE OF THE UNKNOWN SIDE

The cosine of a given angle may not make use of the side which is unknown. It is then necessary to use the other acute angle.

Illustrative example. To what height on a vertical wall will a 38-foot ladder reach, if it makes an angle of 58° with the ground?

In an example like this one, we cannot use either the cosine of 58° or the tangent of 58°, because neither makes use of the two lines, *BC* and *AC*. It is necessary to use the other acute angle, $\angle C$, which in this case is 32°. (Why?) Then we can use the cosine of $\angle C$, *i.e.* $\frac{x}{38}$. Thus we have the equation

$$\frac{x}{38} = .848.$$

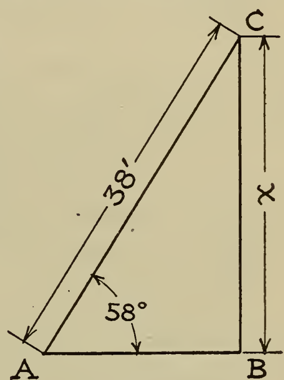


FIG. 84

EXERCISE 43

1. Find *ZY* in right triangle *XYZ*, Fig. 85.

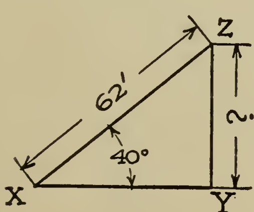


FIG. 85

2. Determine the length of *BC* in right triangle *ABC*, Fig. 86.

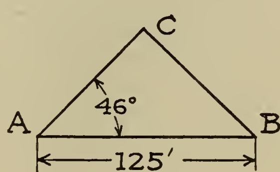


FIG. 86

3. A man travels from O in a direction which is 50° east of a north-south line. How far is he from the north-south line when he has traveled 60 miles from the starting point, O?
4. An aviator, 4200 yd. directly above his own lines, takes the angle of depression of the enemy's battery. What must be the range of the enemy's machine guns to endanger him, if the angle of depression is 29°?

B. FINDING UNKNOWNS BY THE RELATION BETWEEN THE HYPOTENUSE AND THE OTHER TWO SIDES OF A RIGHT TRIANGLE; THAT IS, BY

THE HYPOTENUSE RULE

Section 41. Previous use of the right triangle. We have already seen how right triangles can be used to find unknown distances. If we knew *one* side, and *one* acute angle, we were able to find any other side. Now we come to **another method** of dealing with right triangles; namely, **when two sides are known, but when no acute angle is known.** This method will be illustrated by the following problem:

Illustrative problem. **What is the longest straight line you can draw upon a rectangular blackboard 28 in. wide and 36 in. long?**

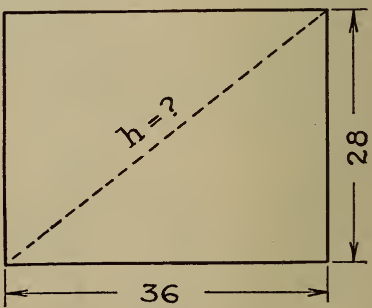


FIG. 87

Evidently the longest straight line is the DIAGONAL of the blackboard, or the HYPOTENUSE OF THE RIGHT TRIANGLE, Fig. 87. Thus, we need to know how to find the hypotenuse of a right triangle when the other two sides are known. This leads to the following:

HYPOTENUSE RULE, OR LAW

which states that **the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides**, or, in the above problem, that

$$h^2 = 36^2 + 28^2.$$

This relation between the sides of a right triangle can be seen from Fig. 88. The base, AB, and altitude, AC, of a right triangle are drawn so that they contain a common unit an integral number of times. AB contains the common unit 4 times and AC contains it 3 times. Then by actual measurement BC will contain the same unit 5 times. By constructing squares on the sides of the triangle, you can see by counting that the sum of the squares on AB and AC is equal to the square on BC.

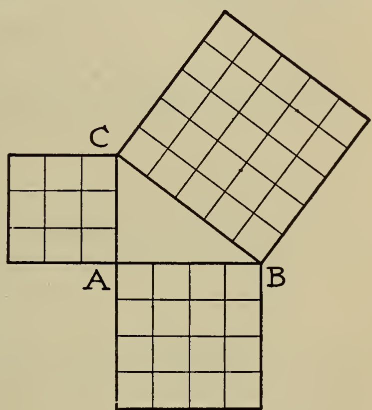


FIG. 88

To test this further, the pupil should construct a right triangle with the base 12 units and the altitude 5 units. Then *actually measure* the hypotenuse, and note whether the square on the hypotenuse is equal to the *sum* of the *squares* of the other two sides. Now we are ready to go back to the problem of finding the longest line that can be drawn upon the blackboard. By making use of the truth which was just studied we have:

$$h^2 = 28^2 + 36^2$$
$$\text{or } h^2 = 784 + 1296$$
$$\text{or } h^2 = 2080$$
$$\text{or } h = \sqrt{2080} = 45.66 \text{ in.}$$

This relation between the sides of a right triangle is more widely used by engineers, carpenters, mechanics, and builders than any other mathematical law. Historical records show that a knowledge of this important relation is nearly as old as civilization itself. It is often called the **Pythagorean Theorem,** because of the fact that **Pythagoras,** a celebrated Greek mathematician, was the first to give a real proof for it.

Note that it was necessary to be able to find the square root of a number in order to find the hypotenuse. That is, we had to solve the equation

$$h^2 = 2080.$$

To solve equations such as this, we must find the square root of each side. A brief review of square root will be helpful at this time.

Section 42. Square root, as you did it in arithmetic. To enable you to use the hypotenuse rule well, you must be skillful in finding the square root of numbers. The illustrative example which follows will review the way you found square root in arithmetic.

Illustrative example. **Find the square root of 200.**

Note the following steps:

(1) The number is separated into periods of two figures each, counting from the decimal point.

(2) You find the greatest square in the left-hand period, and write its square root for the first figure of the root.

(3) Subtract this square from the left-hand period, and with the

```
     200.0000)14.14
     1
  24)100
     96
  281)400
      281
      11900
 2824)11296
        604
```

remainder place the next period for a new dividend. (This is 100 in the example.)

(4) Double the part of the root already found ($2 \times 1 = 2$) for your trial divisor. Divide the dividend, exclusive of the right-hand figure (10) by the trial divisor, 2. Write the quotient obtained, 4, as the next figure of the root and the divisor. Multiply the complete divisor, 24, by the last term of the root, 4. Subtract the product, 96, from the dividend, 100. To the remainder, 4, annex the next period, 00, for a new dividend. Repeat this process until all periods are used, or until any required degree of accuracy is obtained.

EXERCISE 44

PRACTICE IN FINDING SQUARE ROOTS OF NUMBERS

1. 1024
2. 1296
3. 2025
4. 3844
5. 9801
6. 5184
7. 6241
8. 15625
9. 17161
10. 19600
11. 40401
12. 47961
13. 50625
14. 358801
15. 11.6964
16. 372.49
17. 10
18. 50
19. 3
20. 2
21. 8

EXERCISE 45

PROBLEMS BASED ON THE HYPOTENUSE LAW

1. A rectangular schoolroom floor is 32 feet long and 28 feet wide. What is the longest straight line that could be drawn upon the floor?

2. How much walking is saved by cutting diagonally across a rectangular plot of ground which is 25 rods wide and 42 rods long?

3. A tree 100 feet high was broken off by a storm. The top struck the ground 40 feet from the foot of the tree, the broken end remaining on the stump. Find the height of the part standing, assuming the ground to be level. Make a drawing.
4. What is the diagonal of a square whose sides are each 10 in.?
5. Find the side of a square whose diagonal is 20 inches.
6. Two vessels start from the same place, one sailing due northwest at the rate of 12 miles per hour, and the other sailing due southwest at the rate of 16 miles per hour. How far apart are they at the end of 3 hours?
7. The foot of a 36-foot ladder is 13 ft. 6 in. from the wall of a building against which the top is leaning. How high on the wall does the top reach?
8. A rope stretched from the top of a 62-foot pole just reaches the ground 16 feet from the foot of the pole. Find the length of the rope.

SUMMARY OF IMPORTANT POINTS OF CHAPTER VI

1. **To find unknown distances by means of scale drawings is somewhat inaccurate and laborious; to use similar triangles involves two similar triangles, and the method is cumbersome.**
2. **Hence we see the great advantage of the right triangle in finding unknown distances. It can be easily used in two ways:**
3. **The first method is to find the ratio of one side of the triangle to another side. Two different ratios are used: first, the ratio of the "side**

opposite" a given angle to the "side adjacent," the tangent; second, the ratio of the "side adjacent" the given angle to the hypotenuse of the triangle, the cosine.

4. The tangent and cosine of given angles are constant.
5. Hence it has been found convenient to compute the tangents and cosines for all angles from 0° to 90° and compile them in a table; knowing the angle, therefore, one can read the tangent or cosine at once from the table.
6. The second method of using the right triangle to find unknown distances is to use the hypotenuse rule for the relation between the hypotenuse and the other two sides of a triangle, namely: the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

REVIEW EXERCISE 46

In this list of problems you will have to decide for yourself whether to use the tangent or the cosine. Make a drawing for each problem; indicate the parts that you know, and the part you are to find.

1. A flagpole 50 ft. high casts a shadow 80 ft. long. What is the angle of elevation of the sun? What time of year is it?
2. A searchlight on the top of a building is 180 ft. above the street level. Through how many degrees from the horizontal must its beam of light be depressed so that it may fall directly on an object 400 ft. down the street from the base of the building?

3. From the top of a cliff 120 ft. above the surface of the water, the angle of depression of a boat is 20°. How far is it from the top of the cliff to the boat?

4. At a time when the sun was 55° above the horizon, the shadow of a certain building was found to be 98 ft. long. How high is the building?

5. A 40-foot ladder resting against a building makes an angle of 53° with the ground. Find the distance from the foot of the ladder to the building, and the distance from the top of the ladder to the base of the building.

6. A man starts at O and travels in a direction which is 24° west of a north-south line through O, at the rate of 80 miles per day. At the end of 4 days how far north is he from an east-west line through O? How far west is he from a north-south line through O?

7. What direction will a boy be from his starting point if he goes 40 miles due north and then 18 miles due east?

8. The gradient or slope of the railroad which runs up Pike's Peak is, in some places, 18%, *i.e.*, in going 100 ft. horizontally it rises 18 ft. What angle does the road make with the horizontal?

9. A searchlight on the top of a building is 180 ft. above the street level. Through how many degrees from the horizontal must its beam of light be depressed so that it may shine directly on an object 800 ft. down the street from the base of the building? How does your result compare with that obtained in Ex. 2, page 113?

10. From the tenth story of a building the angle of depression of an object on the street level is 30°; from the eighteenth story the angle of depression of the same object is 50°. Find the distance from the object to the base of the building if the second observation point is 80 ft. above the first observation point. Consider the eighteenth story 180 feet above the level of the street.

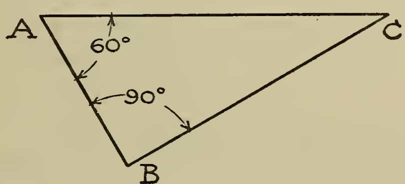


FIG. 89

11. Find AB and BC in Fig. 89, if $AC = 100$ ft.

12. How would you proceed to find the area of the triangle represented in Fig. 90?

FIG. 90

13. Draw a triangle in which AB shall represent 20 ft. and BC shall represent 18 ft. Is it correct to say that tangent $A = \frac{18}{20}$? Why?

14. An aviator observes that the angle of depression of a machine gun nest is 50°. How far is it from a point on the level ground directly below the aviator, if he is 4000 feet high? How far is the aviator from the nest?

CHAPTER VII

HOW TO REPRESENT AND COMPARE QUANTITIES BY MEANS OF STATISTICAL TABLES AND GRAPHS

Section 43. What we have already learned: How to solve equations and to find unknowns. We have now studied how to find unknown values by several methods. First, we have learned how to use the simple equation to find unknown numbers and how to construct practical formulas to aid us (Chapters I, II, and III). Then we learned how to find unknown distances by three methods: *first*, by making scale drawings (Chapter IV); *second*, by using the corresponding sides of similar triangles (Chapter V); and *third*, by using the sides of the right triangle (Chapter VI). In each of these latter methods we have also used equations. Thus you can already see the importance of being able to use the equation skillfully.

Section 44. Need for methods of comparing quantities. The chief aim of mathematics, however, is to help us in comparing quantities and in seeing how quantities are related to each other. For example, in conducting school work, we need to be able to compare the work of one school with another, or of one class with another. We need to be able to represent such facts as the progress of pupils in their studies, to compare the attendance in schools, or the population in cities, etc.

Section 45. Facts of similar kind are called STATISTICS. When we have a great many quantities, like these, all of the same general sort, we speak of them as **statistics**, or **statistical facts**. To illustrate: Suppose you read in the paper that the size of the graduating class in your school last June was 97, and that for each ten years previous the number of graduates had been, 1909, 52; 1910, 57; 1911,

55; 1912, 63; 1913, 70; 1914, 75; 1915, 81; 1916, 83; 1917, 91; 1918, 97. These figures are called statistical facts, or just "**statistics**."

Many illustrations can be found in everyday life of practical statistics: The school marks of children in a class; the number of inhabitants in a list of cities, or in a given city for a number of years; the heights of boys of a certain age; the wealth of countries, or cities, or individual persons; etc. Since we deal with such statistics so commonly it is helpful to have economical methods of expressing and comparing them. In this chapter we shall study these methods.

Section 46. THREE IMPORTANT METHODS OF DEALING WITH QUANTITIES IN MATHEMATICS

There are three methods of representing facts, of comparing them, and of expressing the relationship between them.

(1) The **tabular method**: in which we compile the facts in a table.

(2) The **graphic method**: in which we make a graphic picture of the data.

(3) The **equational** or **formula method**: in which we show relationship between the data by means of an equation.

In this chapter we shall study merely the first two; that is, we shall learn how to make and interpret statistical tables and graphs. Let us study graphic methods first.

I. THE GRAPHIC METHOD OF REPRESENTING QUANTITIES

Section 47. How the Bar Graph is used to picture data. A very common method of representing statistics in newspapers and magazines, in business offices, and in school text-

books, is by means of **BAR GRAPHS.** Figure 91 illustrates the use of the method. It shows by a diagram the marks that were given to 37 pupils on an examination in mathematics. The marks were distributed as follows:

Table 3

2 pupils got A	8 pupils got D
9 pupils got B	4 pupils got F
14 pupils got C	(that is, they failed)

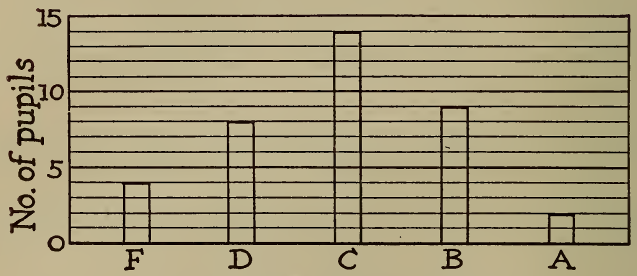


Fig. 91. **Marks received by pupils.**

The teacher, Miss Evans, in sending the marks to the principal, represented them in two ways — first by a graph like Fig. 91 and also arranged in a table like Table 4.

Table 4

Marks received by 37 pupils in a mathematics class	Date: May 27, 1919 No. of pupils who received each mark
A	2
B	9
C	14
D	8
F	4
Total no. of pupils (N) =	37

Notice carefully how she graphed these facts. First she marked off five points on a horizontal line or scale, to represent the marks, A, B, C, D, F. Next she marked off a number of units on the vertical scale which shows the number of pupils who got each mark. Using these two scales, and the units which she selected, she erected lines at A, B, C, D, F tall enough to represent the number of pupils who were given each mark. For example, since 14 pupils got C, the line at C is 14 units long; similarly, for A, B, D, and F.

To make the pictorial effect clearer it is the custom to blacken the lines so that they look as in Fig. 92.

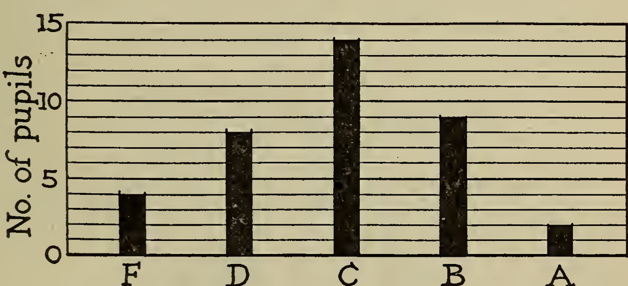


FIG. 92. **Number of pupils who received each mark, A, B, C, D, and F.**

One of the other teachers in the school suggested that the scale need not actually be drawn at the left with the units marked off on it. She said the graph would be easier to read and would tell exactly the same thing if it were made like Fig. 93.

What do you think? What represents the **number of pupils** in each case?

Figures 92 to 95 give illustrations of several kinds of vertical bar graphs. It is very important in reading maga-

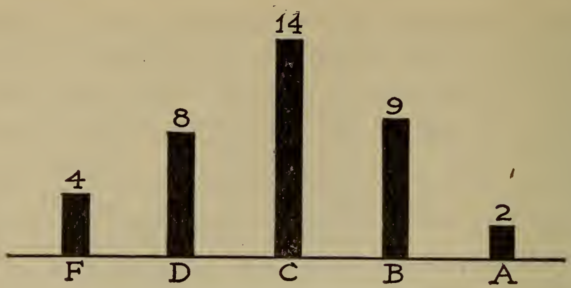
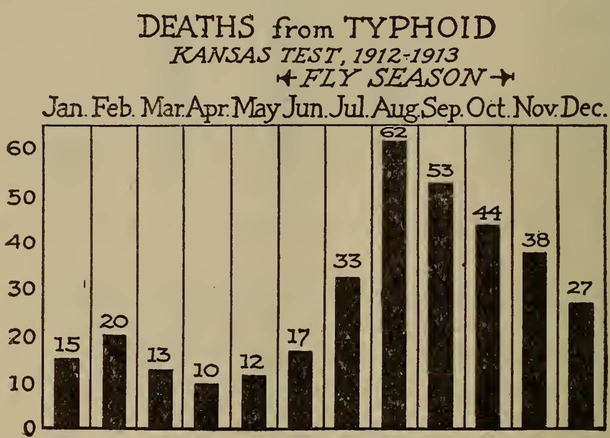


FIG. 93. **Marks received by pupils.**

zines, newspapers, etc., to be able to interpret accurately the meaning of various kinds of graphs. Several are included in this chapter, to give you practice in doing this. Note that they are taken from many kinds of statistics.



Courtesy of International Harvester Co.

FIG. 94

Interpret Fig. 94. What does the number above each bar mean? What has been left off that is included in

Fig. 92? Can you suggest better ways of graphing such facts as these?

Note another kind of graph and kind of **statistics** in Fig. 95. What is the meaning of the decreasing heights of the bars?

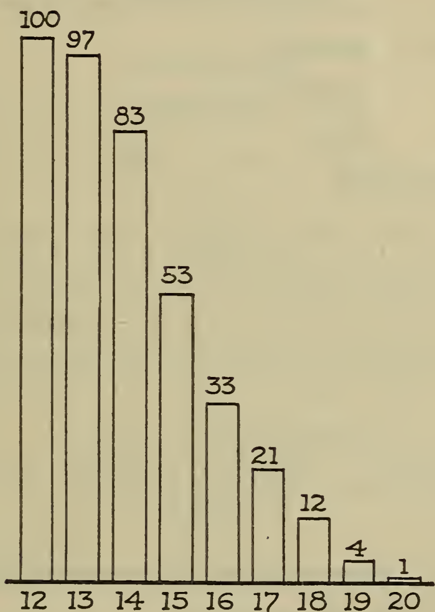


FIG. 95. **Columns represent number of pupils among each hundred beginners who remain in school at each age from 12 to 20.**

Section 48. The use of HORIZONTAL bar graphs to represent quantities. In each of the illustrations which you have been studying, Figs. 91 to 95, the number of pupils or the number of deaths, etc., was represented by **vertical** lines. This is generally done. Sometimes, however, the

number is recorded on the horizonal axis. Figures 96 to 100 show how horizontal bar graphs are used to represent quantities.

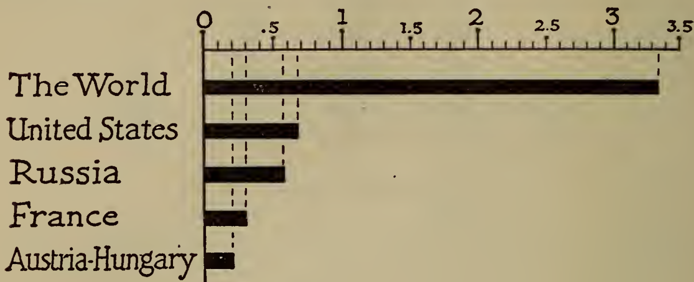
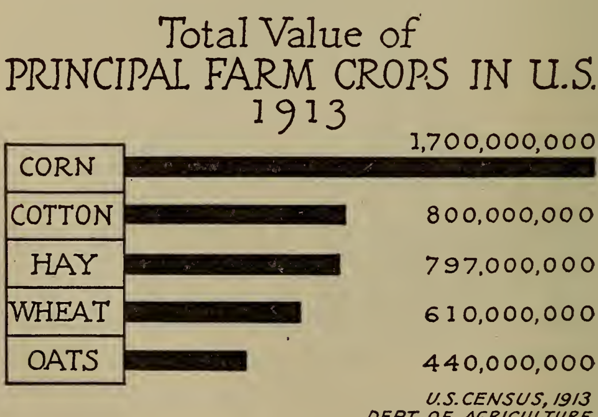


FIG. 96. **The world's average production of wheat 1905–1909, in billions of bushels.**

How could the interpretation of Fig. 96 be made clearer? What do you think about making the "scale" in this form? What improvement on it do you see in the next figure, No. 97?



Courtesy of International Harvester Co.

FIG. 97

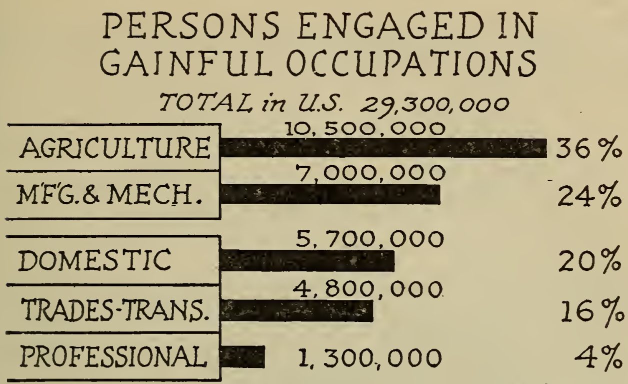


FIG. 98

Interpret Fig. 99 as fully as you can. What is the purpose of making part of the bar graph black and leaving part just in outline?

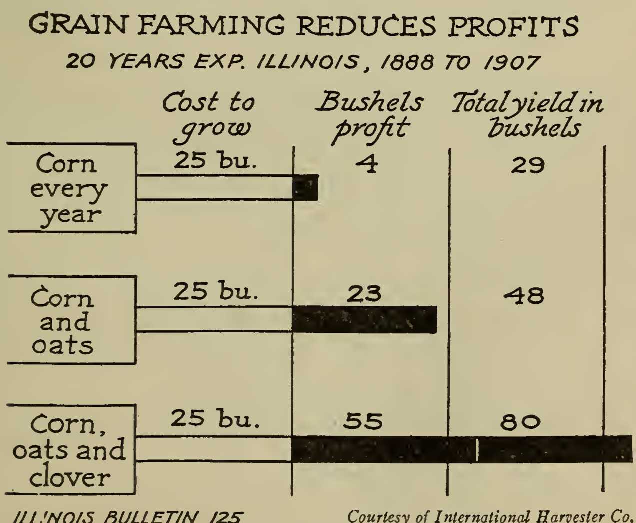
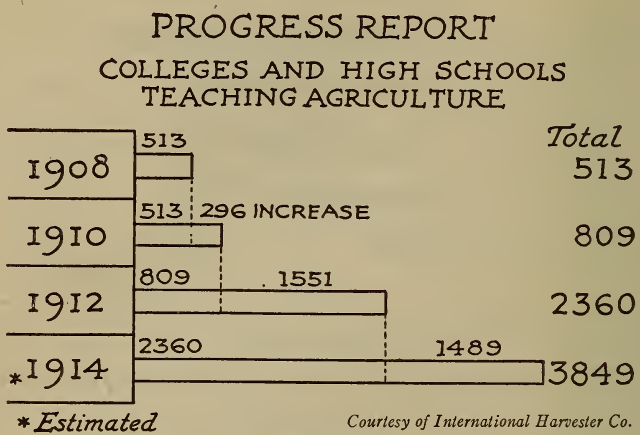


FIG. 99

In what way is Fig. 100 less clearly drawn than Fig. 99? What is the meaning of the * opposite 1914 in Fig. 100?



Courtesy of International Harvester Co.

FIG. 100

Section 49. How a continuous line graph can be used to represent quantities. Sometimes it is convenient to represent the data by a continuous line which joins the top

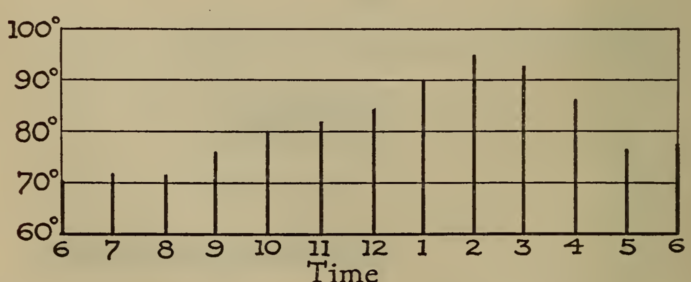


FIG. 101. **Vertical lines to represent temperature at different hours of the day. (Data in Table 5.)**

points of the bars or lines in figures like Figs. 102, 103, and 104. For example, suppose you wished to tell some one what the temperature was in Chicago at each hour of a certain day from 6 A.M. to 6 P.M. It could be done as in Fig. 101.

The graph can be read even more easily, however, if it appears as a **LINE GRAPH** like Fig. 102.

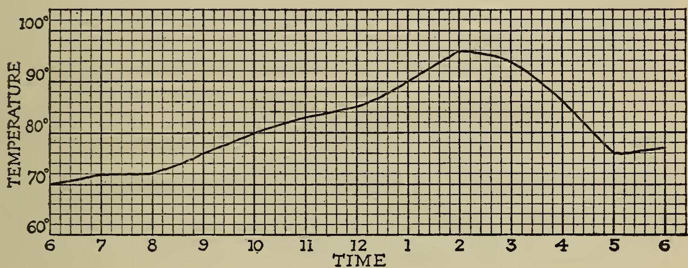


FIG. 102. **Graphic representation of temperatures at various hours of the day.**

Note that the **horizontal line**, or scale, shows the hours. This is the **time scale.** Each large space represents one hour. The **vertical line**, or **scale**, shows the temperature; on *this scale* each large space represents 10°, or each small space represents 2°. Suppose we wanted to read from the graph what the temperature was at 11 o'clock. We find it by looking along the *time line*, or *time axis*, until we come to the point marked 11 o'clock. We then look up to the line of the graph. In this case we have to go up to a point $11\frac{1}{2}$ small spaces above the 11 o'clock point. By looking back, to the left, to the vertical or temperature scale, we see that any point on this horizontal line stands for 83°. Hence the graph shows that at 11 o'clock the temperature was 83°.

Section 50. A continuous line graph makes it easier to INTERPOLATE: to find values between those that are actually measured and recorded. Do you see any other real difference between these two kinds of graphs, the continuous line graph in Fig. 102 and the bar diagram of Fig. 101? What kinds of facts can you get from the graph of Fig. 102 that you cannot get as easily from Table 5? How can you tell about what the temperature was at 8.30 A.M.? Could you tell from Fig. 101? What would you have to do to Fig. 102 in order to tell?

II. THE TABULAR METHOD: MAKING A TABLE WHICH WILL REPRESENT THE QUANTITIES

Section 51. There is a second way in which such facts as those in Fig. 102 can be conveniently represented. This is by making a table like Table 5.

TABLE 5

AN ILLUSTRATION TO SHOW THE WAY TO TABULATE TEMPERATURE AT DIFFERENT HOURS OF THE DAY

	A.M.							P.M.					
Hour	6	7	8	9	10	11	12	1	2	3	4	5	6
Temperature	70	72	72	76	80	83	85	90	96	94	86	76	77

This method, which we shall call the **TABULAR METHOD,** shows the way the temperature changes at different hours of the day. To understand the table, however, requires much more effort on the part of the reader than is required to understand the first method, which is shown above. This pictorial or **graphic method** shows all that the **tabular method** shows, and has the advantage of being more easily interpreted.

The following questions will help you compare the

graphic and **tabular** methods of representing the relation between two numbers.

EXERCISE 47

In order to answer each of these questions, refer to the data of Table 5 and Fig. 102.

1. Find, both from the table and from the graph, Fig. 102, the highest temperature.
2. What was the lowest temperature? Which shows this the more easily, the table or the graph?
3. Between what hours did the temperature change the most rapidly?
4. About what do you think the temperature was at 9.30 A.M.?
5. Between what hours did the temperature change the least?
6. What might explain the rapid fall in temperature between 4 P.M. and 5 P.M.?

After answering the questions, are you not convinced that the **graphic method gives the information** which the reader may desire **much more quickly and easily than the tabular method**? The fact that this is true has brought about a very wide use of graphic methods in all kinds of business and industry. Nearly every newspaper and magazine contains "*graphs*" of some kind. Your teacher will be glad to have you bring to class any graphs you may find in the newspapers or magazines.

In the following exercise you will get practice in graphing various kinds of statistical facts.

EXERCISE 48

PRACTICE IN REPRESENTING FACTS GRAPHICALLY

1. Make a bar graph representing the cost of education per pupil in Grand Rapids for the years 1906–1916.

1906	\$38	1910	\$46	1914	\$51
1907	39	1911	49	1915	53
1908	44	1912	49	1916	55
1909	45	1913	50		

2. Represent graphically the results of a test which a teacher gave in freshman English. In her class of 29 pupils, 4 failed (F), 12 got C, 5 got D, 6 got B, and 2 got A.

3. In a newspaper the graph shown in Fig. 103 was printed. It gives the prices of wheat, per bushel, from August 5 to August 10. What was the price on August 5? On August 7? On August 9?

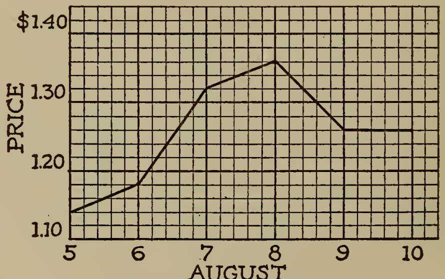


FIG. 103. **Graphic representation of prices of wheat on various days of August, 1916.**

4. When was the price the highest? the lowest? Between what dates did the price change most? change least?

5. Each large space on the vertical scale represents how many cents? What is measured along the horizontal scale? What is the unit used on this scale?
6. In 1900 40.5% of the total population of the United States were living in cities of 2500 or more. In 1910, 46.3% of the inhabitants lived in such cities. Represent these facts by means of a bar graph.
7. Draw a line graph to show the increase in size of the graduating class of a high school whose graduating classes were as follows:

1908	56	1911	79	1914	110	1917	131
1909	72	1912	84	1915	117	1918	139
1910	73	1913	101	1916	124		

8. The table below gives the earnings of a book agent for the latter part of July, 1915. Show the same thing graphically.

TABLE 6

Date	19	20	21	22	23	24	25	26	27	28
Earnings-$	2.00	3.50	4.00	5.00	7.00	4.50	3.00	8.00	7.50	9.00

SUGGESTION. Represent time on the horizontal scale, and earnings on vertical scale.

SUMMARY OF IMPORTANT ASPECTS OF GRAPHIC REPRESENTATION

Section 52. In the study of the previous examples, the following important aspects of graphic representation should be noted:

1. Graphs always show the relation between two changing quantities; for example, they show

the relation between the temperature and the time of day.

2. Two rectangular *axes* are drawn. One of the **changing quantities** is measured on the horizontal axis; the other **changing quantity** is measured on the vertical axis.

3. These axes, or reference lines, are **scales**, marked off in a series of **units**. Thus, as in our illustrative examples, the horizontal axis may be a **time scale**, marked off into **units** of one hour each, and the vertical axis may be a temperature **scale**, marked off into units of **two degrees** each.

4. In making a graph one must choose units very carefully in order to be able to get all the information on the graph, and yet make it stand out as clearly as possible.

HOW TO COMPARE SUCCESSIVE RELATED FACTS

Section 53. How graphs are used to compare successive or related facts. There are very practical applications of the sort of graphing work that you have been studying. An illustration is found in graphing the progress that pupils make in school. The following example illustrates it.

Illustrative example. The pupils in a certain mathematics class took a three-minute practice test every day. The object was to see how many examples in evaluation (like those in Chapter III) each pupil could solve correctly in the three-minute period. The scores of one of the

pupils, Frank Johnson, for fourteen successive days, were as follows:

TABLE 7

Jan. 14th	1 example	Jan. 22d	8 examples
Jan. 15th	5 examples	Jan. 23d	10 examples
Jan. 16th	7 examples	Jan. 24th	10 examples
Jan. 17th	8 examples	Jan. 25th	9 examples
Jan. 18th	6 examples	Jan. 28th	11 examples
Jan. 21st	9 examples	Jan. 29th	11 examples

He made a blank form like that in Fig. 104, and each day plotted a point to represent graphically his score. He also connected the points that he obtained by a continuous line graph, as in Fig. 104.

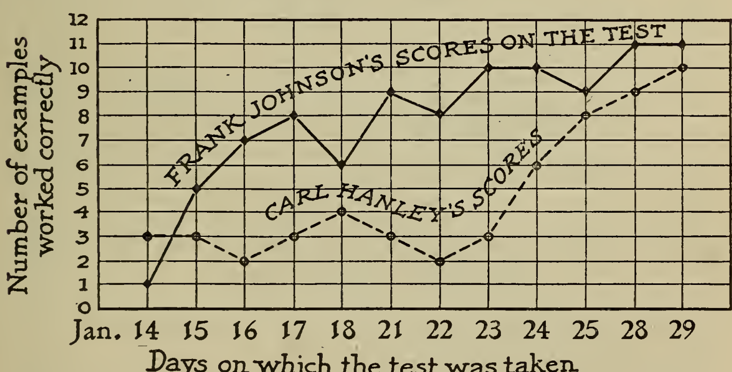


FIG. 104. **Comparison of two pupils' work in a mathematics test on 12 successive school days.**

How would you interpret this graph? Tell all the important things you can see in it about Frank's ability to solve simple equations. What would explain the drop in the graph on January 18th, 22d, and 25th? What do you

think was true of Frank's skill in doing equations by January 29th?

Carl Hanley's scores on the same days, in the same class, are given in the same chart, Fig. 104. Compare his ability to solve equations rapidly and accurately with that of Frank. Why do you think his curve rises rapidly on January 24th and after? Interpret the graph carefully.

Figures 105 and 106 illustrate the use of the line graph.

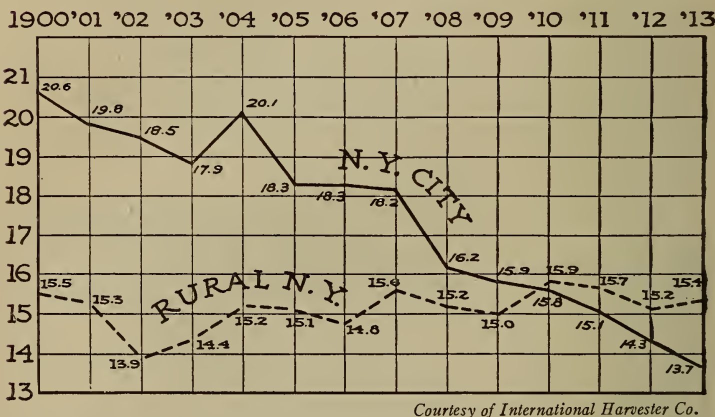


FIG. 105. **Death rate in New York City compared with that in rural New York in 14 successive years.**

Why does New York City's curve fall off while rural New York's remains fairly level?

What is the significance of the rising and falling lines? What do vertical distances represent in this figure? What do horizontal distances represent?

Can you make a single table which would show all of the facts of Fig. 104? How many columns will be required?

Use the blank form in Fig. 107 to plot your own score on the practice exercise on page 48.

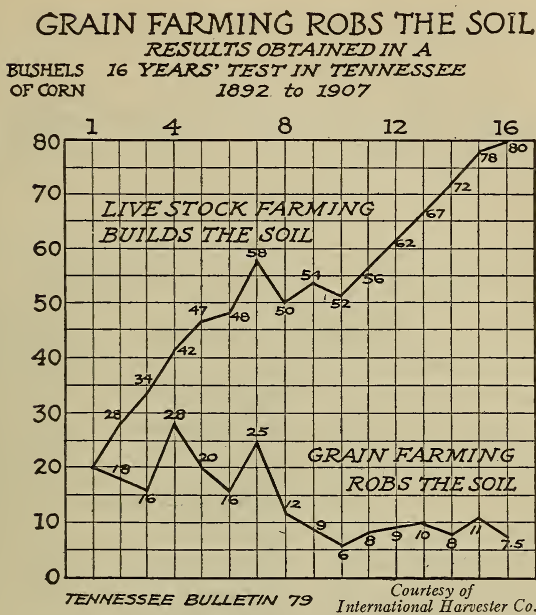


FIG. 106. Comparison of number of bushels of corn obtained per acre under live stock farming with the number obtained under grain farming.

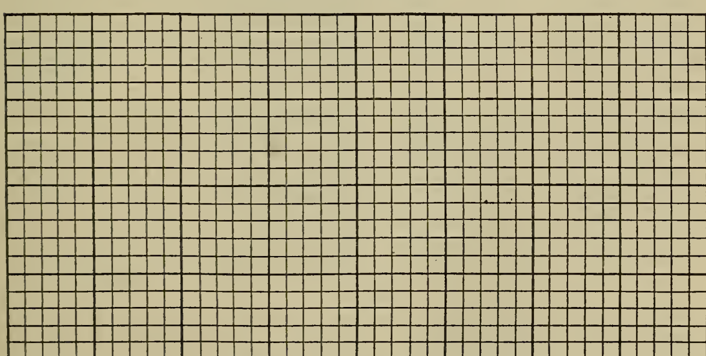


FIG. 107. Plot your scores here.

HOW TO ARRANGE AND COMPARE SCATTERED DATA

I. PREPARING A TABLE

Section 54. How to arrange data systematically in a table. It is very **important to be able to tabulate facts** clearly and systematically. You have already had a little practice in tabulating quantities which change together.

TABLE 8

Miss Hitchcock's Class		Mr. Davis's Class	
Pupils	No. ex. right in 3 minutes	Pupils	No. ex. right in 3 minutes
Adams, Ada	17	Barwick, Dorothy	3
Albright, J. H.	11	Barton, John	15
Bass, Dan	13	Brown, William	13
Brownell, Bessie	10	Bryant, George	19
Carlson, Anna	18	Bruner, C. H.	17
Crowther, Jas.	4	Darling, Jane	14
Dawes, Janette	9	Dennison, Chas.	7
Evans, Isabel	11	Erby, Claude	9
Finch, Geo.	12	Evers, Charlotte	14
Ford, Wm.	11	Fitzpatrick, J. M.	15
Harris, David	9	Haffner, Henry	5
Herrick, H. E.	8	Haffner, Margaret	10
Hogan, John	6	Hall, Marion	14
Johnson, Emma	19	Irwin, Jean	20
Lanterman, Anne	16	Isaacs, Walter	16
Lowenthal, Louis	15	Kerby, Harold	11
Manning, Fred	10	Lowe, Mary	14
Marston, Mary	11	Macpherson, Edythe	13
McMurray, Mabel	12	Marcy, Leslie	12
Mendenhall, Carl	15	Morton, Aaron	8
Metz, Pauline	14	Norton, Beatrice	6
Owens, Edward	12	Parker, Suzanne	5
Ranney, Geo.	5	Parsons, Julia	13
Reed, Katherine	3	Reynolds, Oscar	14
Smith, John	14	Rhoads, J. E.	15
Wright, Evelyn	13	Rice, Marie	4
Wright, Betty	11	Royston, Ralph	2

There are many occasions in which we wish to compare scattered data. In such it is necessary to arrange the data in a table. An illustration can be given.

Illustrative example. Miss Hitchcock's class and Mr. Davis's class both took a simple equations test. The pupils in the two classes made the scores shown in Table 8 on the opposite page.

Which class did better work? Why is it difficult to compare the work done by the two classes? Is it not because the **test results are not arranged in orderly fashion**? For example, in which class is the pupil who got the very best scores? the very poorest? Were there more very good scores in Miss Hitchcock's class or in Mr. Davis's? To answer such questions what do you have to do, as the facts are arranged in the above lists?

Now note how much more easily you can answer the questions from Table 9. The children and their **scores are now arranged in an orderly way.** They are arranged so that the pupil who made the best score is put first, the next best second, and so on through the list. The poorest score is put last. **This is called RANKING** the pupils' scores, or **placing them in RANK ORDER.**

From a glance at Table 9 on page 136 you can see that the best pupil in Miss Hitchcock's class got 19, while the best in Mr. Davis's got 20. On the other hand, the poorest pupil, 2, is in Mr. Davis's class. The pupils are now arranged in convenient order in the two classes.

Importance of grouping similar scores in a FREQUENCY TABLE. Even now, though, it is difficult to tell which did the superior work. The figures are still too scattered.

Table 9

THE SCORES OF THE PUPILS IN THE TWO CLASSES ARRANGED IN RANK ORDER

Miss Hitchcock's Class		Mr. Davis's Class	
Pupils	Scores	Pupils	Scores
Johnson, Emma	19	Irwin, Jean	20
Carlson, Anna	18	Bryant, George	19
Adams, Ada	17	Bruner, C. H.	17
Lanterman, Anne	16	Isaacs, Walter	16
Lowenthal, Louis	15	Fitzpatrick, J. M.	15
Mendenhall, Carl	15	Rhoads, J. E.	15
Metz, Pauline	14	Barton, John	15
Smith, John	14	Lowe, Mary	14
Bass, Dan	13	Reynolds, Oscar	14
Wright, Evelyn	13	Darling, Jane	14
Finch, George	12	Evers, Charlotte	14
McMurray, Mabel	12	Hall, Marion	14
Owens, Edward	12	Macpherson, Edythe	13
Albright, J. H.	11	Parsons, Julia	13
Evans, Isabel	11	Brown, Wm.	13
Ford, Wm.	11	Marcy, Leslie	12
Marston, Mary	11	Kerby, Harold	11
Wright, Betty	11	Haffner, Margaret	10
Brownell, Bessie	10	Erby, Claude	9
Manning, Fred	10	Morton, Aaron	8
Dawes, Janette	9	Dennison, Chas.	7
Harris, David	9	Norton, Beatrice	6
Herrick, H. E.	8	Parker, Suzanne	5
Hogan, John	6	Haffner, Henry	5
Ranney, Geo.	5	Rice, Marie	4
Crowther, Jas.	4	Barwick, Dorothy	3
Reed, Katherine	3	Royston, Ralph	2
Total	27 pupils	Total	27 pupils

Is there any way we can further combine or group them to help us decide more easily? Is there any single score in one class that we can compare with the same score in the other? For example, how many pupils did 11 problems? 5 in Miss Hitchcock's class and 1 in Mr. Davis's.

On the other hand, 5 did 14 right in Mr. Davis's and only 2 in Miss Hitchcock's.

This shows us that we can save time and make the comparison clearer by **grouping together all pupils who did the same number of examples.** Furthermore, we do not need the names of the pupils, to make our comparison. Table 10 shows the same data grouped in convenient form.

TABLE 10

Test Scores Made by Pupils	No. of Pupils Who Made Each Score	
	In Miss H's Class	In Mr. D's Class
20		1
19	1	1
18	1	0
17	1	1
16	1	2
15	2	3
14	2	5
13	2	2
12	3	1
11	5	1
10	2	1
9	2	1
8	1	1
7		1
6	1	1
5	1	2
4	1	1
3	1	1
2		1

The table gives the frequency with which each score, *i.e.* each number of examples, was made. For this reason, it is called a **FREQUENCY TABLE.** Notice that the scores **are still arranged conveniently in RANK ORDER.**

Can you tell any more easily, now, which class did the better work on the test? **Is there any single score** that we could use to compare the work of the two classes? How would it do to compare the number in the two classes who solved 11 problems? If we do, Miss Hitchcock's class is best, for 5 in her class did 11 problems, and only 1 in the other class. Suppose we took 14 problems — then Mr. Davis's is best, 5 pupils making this score in his class, against 2 in Miss Hitchcock's.

Bar graphs make the comparison clearer. A clearer comparison can be made if we make a graph of each set of scores. Fig. 108 shows this method.

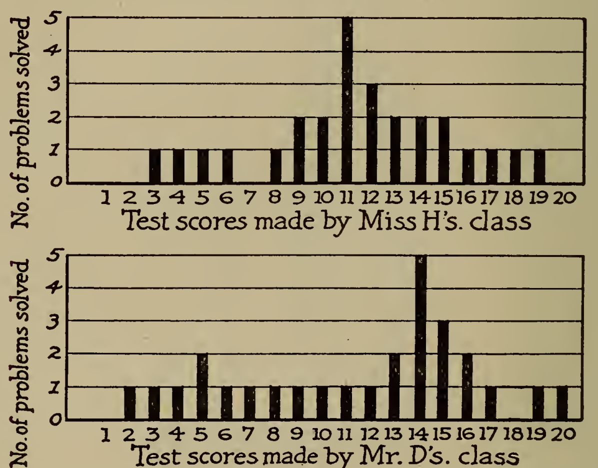


FIG. 108. **Graphic comparison of scores made by two classes in a mathematics test.**

Can you tell now which of the two classes is superior? Why do you think Mr. Davis's class is? Do the scores seem to be higher, as a general rule; that is, are they distributed farther along toward the right end of the scale?

How does the **height of the bars** in the graphs help you decide? Does the tall bar at 14 examples in Mr. Davis's class influence your decision?

COMPARING DISTRIBUTIONS OF DATA BY MEANS OF SINGLE NUMBERS

Section 55. The use of a single number: The mode, the median, and the average. When we work with data that are so scattered and have to do with so many different values, we need single numbers which will tell us the **tendency of the data to center around particular values. This is called the CENTRAL TENDENCY of the data.** Thus, we are interested to find whether there are numbers that represent the central tendency of all the measures. There are three numbers that do this very well: (1) **the mode or commonest number**, and (2) **the median or middle number**, and (3) **the average**. We shall discuss these next.

Section 56. First: THE MODE: the commonest number. This tallest bar at 14 examples, in the scores of Mr. Davis's class, tells you that the score, 14, was made more frequently than any other. It is the **commonest score**. People who work with scattered statistics call this the **MODE** (the word comes from the French word, "*la mode*," meaning "*the fashion*").

The **mode** of the scores made by Miss Hitchcock's class is 11. This is 3 examples less than the mode of the scores in Mr. Davis's class, and tends to show that the latter is superior. What is the **mode** of the marks made by the pupils represented in Fig. 91? What of the data in Fig. 94? What is the **mode** in Figs. 95 and 98? **Just what does this one figure tell you about each graph or each distribution?**

Do you think this single number represents the tendency of the measures to be gathered at one place on the scale? Thus one can use the mode to help compare two sets of data. The mode is the easiest single number to find, for you simply determine from the tallest bar of the graph which number occurred most frequently.

Section 57. Second: THE MEDIAN: the middle number. There is another number that is easily found and that represents all of the numbers in the list rather well. This is **the middle number.** It is called **THE MEDIAN.** It can be found by merely **counting** from one end of the list. Why? For example, the **median** score made by Miss Hitchcock's class is 11, because there are 27 pupils in the class and the **middle one** is the thirteenth. This thirteenth pupil from either end is one of the 5 pupils who made scores of 11. So the **median** score is 11. The **median** in Mr. Davis's class is 14. Why? How does this help you to decide which class did the better work? All of our work so far shows that Mr. Davis's class did.

Section 58. Third: THE AVERAGE: another single number which represents all of the data. In the elementary school you have already learned how to find the **average** of a series of numbers. For example, how do you find the **average** of the scores of the 27 pupils in Miss Hitchcock's class? of those in Mr. Davis's class? What do you total? What do you divide by? If you let each fact or measure be represented by m, and the total number of measures by n, can you make a formula for the value of the **average**? What is the **average** in each case? Which is the higher? What does this tell you about the work of the two classes? How does it compare with what the **median** and the **mode** each told you?

The median and the mode and the average are measures of "central tendency." So we can say that the most typical or **representative** score in Mr. Davis's class was 13, while in Miss Hitchcock's class it was 11.

What is the median of the distribution in Fig. 93? What is the average in Fig. 108? In what way are these measures of central tendency?

THE GENERAL SHAPE OF THE DISTRIBUTION OF MOST HUMAN TRAITS

Section 59. How scores are generally distributed. Turn back to Fig. 91. Study the frequency with which each mark, A, B, C, D, and F, occurred; this frequency gives a **graph, or curve, of a particular shape.** Now study the shape of the graphs in Fig. 108. Do you notice any resemblance between them?

TABLE 11

Heights in centimeters	126	128	130	132	134	136	138	140	142	144	146	148	150	152	154	156	158	160
Frequency, i.e., no. boys	1	5	14	24	39	58	96	120	150	142	123	88	63	36	23	12	5	1

In Table 11, the heights of 12-year-old boys are given as they were actually measured in a certain school. The

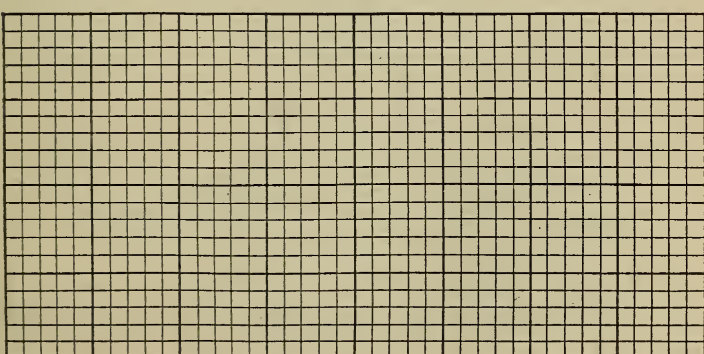


FIG. 109

heights are measured in centimeters. Plot a graph in the blank form in Fig. 109 to represent these facts.

In exactly what way does this graph resemble the others? In Figs. 110 to 113 measurements of four other human traits are graphed. What fact do you notice of the shape of these graphs that is like the others?

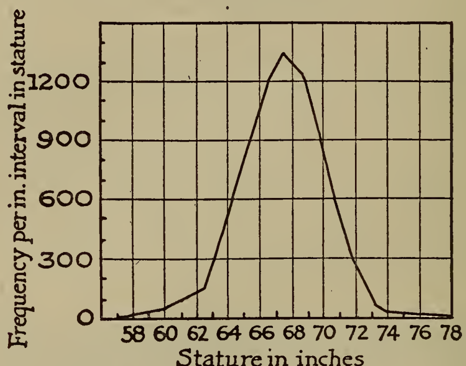


FIG. 110. **Heights of 8585 men grouped in inch intervals.**

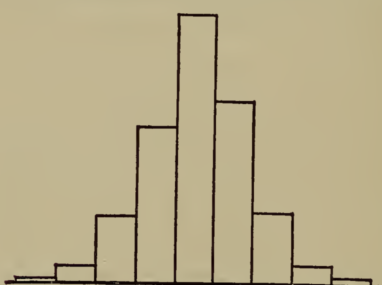


FIG. 111. **How 904 children distributed in intelligence from very bright to very dull.**

Section 60. These graphs all resemble each other in one way. **The heights of the lines or bars at the extreme low and high ends of the curve are very short. Those nearer**

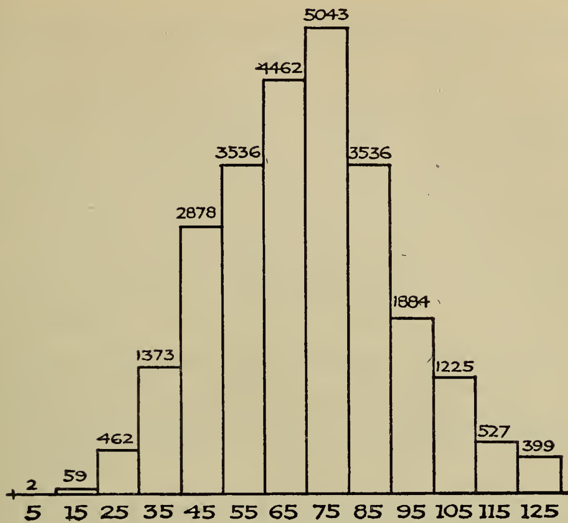


FIG. 112. Number of pupils writing at each speed from 0 to 9 letters per minute to 120 to 129 letters per minute. Data for 25,387 pupils in four upper grades.

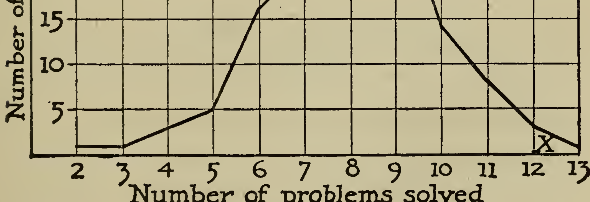


FIG. 113. Number of pupils who solved various numbers of algebra problems.

the middle are longer and that nearest the very middle tends to be longest. We interpret this fact by saying that there are always more average or middle measures than extreme ones. Notice how true it is of every graph of facts that have to do with human beings. More pupils got average marks (Fig. 91) than very good or very poor. There were more boys of average height (that is, about 142 centimeters) than there were very tall or very short (Fig. 110). More fifth-grade pupils in Cleveland could write at an average rate of about 75 words a minute than at a very rapid rate, like 125 words, or a very slow one, like 10 to 25 words per minute. More pupils could solve about 8 examples in algebra than any other number. So it is with all measurements of human traits. Many careful measurements have been made of both physical and mental abilities during the past 100 years and they all show the same kind of graph.

Section 61. You have seen that there are always more "average" measures than extreme ones. We have already used single numbers to characterize this central tendency. Now we see that the graphic representation of the facts confirms the conclusions which we obtained by comparing statistics by single numbers.

SUMMARY OF CHAPTER

The important points of this chapter are as follows:

1. **Facts of similar kind are called statistics.**
2. **There are three methods of dealing with quantities in mathematics: (1) the tabular method; (2) the graphic method; (3) the equational or formula method.**

3. Horizontal and vertical BAR GRAPHS are used to picture statistical facts.

4. Continuous line graphs are also used and enable us to INTERPOLATE. There is an important summary of graphic methods on page 129.

5. Statistical facts can also be represented by tabulating them in a table. It is not so easy to compare them by this method as by the *graphic* method.

6. *Line graphs* are used to compare successive facts.

7. Scattered statistical facts can be grouped and compared clearly by arranging them in RANK ORDER.

8. It is still clearer to group the quantities in a FREQUENCY TABLE.

9. The frequencies may be clearly shown by plotting them as vertical or horizontal bar graphs.

10. When we plot such FREQUENCY DISTRIBUTIONS the bars commonly are longest (that is, the frequencies are greatest) near the middle or average part of the scale.

11. Hence we can compare two or more frequency distributions by computing an average to show the CENTRAL TENDENCY.

12. There are three important measures of CENTRAL TENDENCY: (1) the MODE, i.e., the most frequent number; (2) the MEDIAN, i.e., the middle number; (3) the AVERAGE, i.e., the sum of the measures divided by the number of measures.

13. With measurements of traits of human beings, the average or central measurements always occur most frequently. The measurements occur less frequently as they become very small or very large. We interpret this as follows: "More ordinary people occur in the world than unusual ones."

REVIEW EXERCISE 48 *a*

Two classes in a certain school took the same test in Simple Equations. The results are given here.

Mr. Evans's Class		Miss Brown's Class	
Pupil	No. examples right	Pupil	No. examples right
A	1	A	2
B	4	B	4
C	5	C	5
D	5	D	3
E	3	E	4
F	6	F	5
G	4	G	4
H	3	H	7
I	5	I	5
J	4	J	7
K	3	K	0
L	7	L	3
M	2	M	4
N	5	N	2
O	2	O	4
P	4	P	4
Q	8	Q	3
R	4	R	5
S	5	S	4
T	1	T	6
U	6	U	1
V	2	V	6
W	4	W	6
		X	2
		Y	3

Which class did the better on the test? What is the median number of *rights* in each class? Find the *average*. Represent the two tables graphically.

CHAPTER VIII

HOW TO REPRESENT AND DETERMINE THE RELATIONSHIP BETWEEN QUANTITIES THAT CHANGE TOGETHER

Section 62. In mathematics we deal with two distinctly different kinds of graphs. In Chapter VII we tabulated and graphed data in which there was not necessarily a causal relation between the two quantities; for example, the temperature and the hour of the day. This is obviously true of the mere graphic representation of statistical facts like populations, school marks, attendance, wealth, production, etc. For this kind of graph no equation or formula can be made. We speak of this kind of graph as a **STATISTICAL GRAPH.**

Section 63. As we go about our daily work, we commonly deal with quantities which change together. For example, the cost of a railroad ticket *changes* as the number of miles you travel *changes*; that is, the *cost* and the *distance change together*. Or, the *distance* traveled by an autoist, if he goes at the rate of, say, 20 miles per hour, *changes* as the number of hours which he travels *changes*; that is, the *distance* and the *time change together*. As a third illustration, suppose you wanted to make a trip of 100 miles. We know that the time required will change with or be determined by the way in which the rate changes; that is, the *time* and the *rate change together*.

Section 64. The fact that we are always dealing with situations of this kind makes it necessary for us to know how to *represent and determine these quantities which change together*, or which are RELATED in some definite way. Mathematics shows us how to describe or express them. In fact, it is the **chief aim of mathematics** to help you to see how quantities are related to each other and to help you to determine their values.

Section 65. The three methods of representing relationship. You learned in Chapter VII that there were three methods of representing quantities, of comparing them, and of determining the relationship between them. These were: (1) **The graphic method,** (2) **the tabular method,** and (3) **the equational or formula method.** In the problems just taken up in Chapter VII you used the tabular method and the graphic method of representing and comparing facts. In those cases there was no relationship expressed and hence no possibility of using the equational or formula method.

In this chapter, however, we shall deal with facts which are related and learn how to use all **three methods** of expressing and determining the relationship between them. A good illustration is that of the relationship between (1) the **time** a railroad train travels and (2) the **distance** the train travels. Let us take an example.

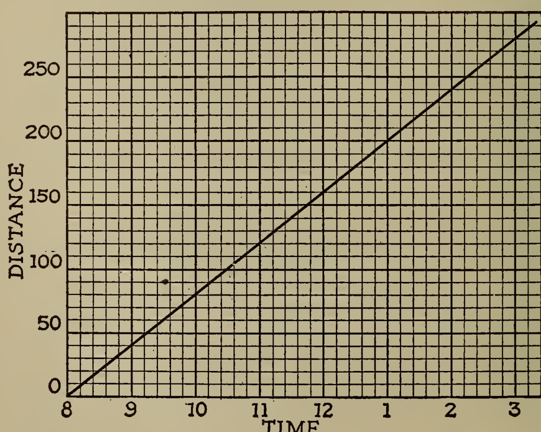


FIG. 114. **The line shows relationship between time spent and distance traveled.**

> **Illustrative example.** An east-bound train, running at 40 miles per hour, left Chicago at 8 A.M. Show from the graph, Fig. 114, how far the train was from Chicago at 10 A.M.; at 11 A.M.; at 11.30 A.M.; at 2 P.M. At what time was the train 100 miles from Chicago? 200 miles?
> In Fig. 114 how many miles does each small space represent? How many hours does each large space equal?

Table 12 also shows the relationship between (1) the *time* the train traveled and (2) the *distance* it went.

TABLE 12

At a given time (hrs.)	1	2	3	4	5	6	7
the distance was (mi.)	40	80	120	160	200	240	280

Second illustrative example. Another good illustration of the graphic method of representing relationship is that shown in Fig. 115.

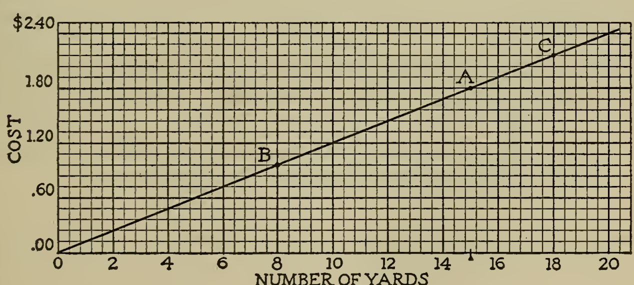


FIG. 115. **The line shows the relationship between the number of yards of cloth purchased and the total cost.**

Figure 115 is a price graph which shows the cost of *any* number of yards of cloth at 12 cents per yard. From it we can find the cost of *any number* of yards. For example, the cost of 15 yd. is found by finding the point on the horizontal scale which stands for 15 yd., then by finding the point on the cost line directly above this point. This appears on the cost line as point A. Now, to find the cost of 15 yd. we find the point on the cost axis, horizontally opposite the point A which already stands for 15 yd. The cost proves to be \$1.80. Thus we see that point A stands both for 15 yd. and for \$1.80. In the same way the point B shows that 8 yd. on the horizontal scale *corresponds* to 96¢ on the vertical scale. The point C shows that 18 yd. on the horizontal scale corresponds to \$2.26 on the vertical scale.

Table 13 shows the same relationship but not so clearly.

TABLE 13

If the no. of yards is	1	2	3	4	5	6	7	8	9	10	11	12	13
the total cost is (\$)	.12	.24	.36	.48	.60	.72	.84	.96	1.08	1.20	1.32	1.44	1.56

THE FORMULA OR EQUATIONAL METHOD OF EXPRESSING AND DETERMINING RELATIONSHIP

Section 66. The picture of relationship. Now let us apply the formula method to the second illustrative example given in the preceding section. Make a formula for the cost of any number of yards of cloth at 12¢ per yard.

Note that the graph and the table and this formula which you have just made tell exactly the same thing. **The graph tells the relation** between the cost and the number of yards purchased **more clearly** because it presents it to the eye as a picture. To tell from the graph the cost of any particular number of yards requires only a glance; to tell from the formula or equation,

$$C = .12\,n,$$

requires that we substitute some particular value of n in the equation and then that we find the value of C.

EXERCISE 49

LET US GET MORE PRACTICE IN USING THESE THREE METHODS

1. Draw a graph showing the price of any number of pounds of beans at 9 cents a pound. From it find the cost of $5\frac{1}{2}$ pounds; of 12 pounds.
2. Now, write a *formula* which represents the cost of any number of pounds at 9 ¢ a pound. Note that *the graph and the formula tell the same thing.*
3. Draw a graph for the cost of a railroad ticket at 3 ¢ a mile.
4. If $c = .03\,m$ is used as the equation for the cost of any railroad ticket at 3 ¢ a mile, show that by letting m have particular values, such as 2, 3, 7, 10, etc., we get values for c, from which we can make the graph.
5. A number of rectangles have the same base, 5 in. Write an equation for the area of any rectangle which has a 5-inch base. (Use h for the altitude, or height.)

6. Draw a graph for the area of any rectangle whose base is 5 in. by using the equation you got in Example 8. (Let h have particular values, such as 2, 3, 4, 7, 10, and find the corresponding area, in each case.)

7. A west-bound train leaves Chicago at 7 A.M., going 30 miles per hour. Show graphically its progress until 4 P.M.

8. Using $d = 30\,t$ for the equation of the train in Example 7, show that the graph could have been made from the results obtained by letting t have particular values.

9. The movement of a train is described by the equation $d = 25\,t$. Draw a graph showing the same thing.

10. A boy joined a club which charged an initiation fee of 25 cents. His dues were 10 cents each month. Draw a graph to show how much he had spent at the end of any number of months.

11. What formula or equation will represent the same thing as the graph in Example 10?

I. VARIABLES: Quantities which are continually varying.

II. CONSTANTS: Quantities which are always fixed or unchanging.

Section 67. In all the examples which you have just solved graphically there have been **changing or varying quantities**; for example, in the graph of the motion of a train, the *distance and the time* **vary** as the train moves along its trip; or in any cost graph the *cost* **varies**

(that is, increases and decreases) *as the number of articles* **varies.**

But in these examples, **some of the quantities do not change or vary.** To illustrate: the *rate* of the train (as in Example 7, 30 miles per hour) remains **fixed,** or **constant,** as the train moves along; and the price per unit of any article (for example, cloth at 12¢ per yard) remains fixed or constant in any particular example.

Thus, in any problem we may have two kinds of quantities: *first,* those that change or **vary**; and *second,* those that remain fixed or **constant.** We call them, respectively, VARIABLES AND CONSTANTS. For example, in the formula for the area of any rectangle whose base is 4 units, $A = 4h$, it is clear that A and h are *variables*, and that the base, 4, remains *constant.* In other words, if h is 2, then A is 8; if h is 3, then A is 12; if h is 7, then A is 28, etc. Thus, h can change, but as it changes, A also changes, since A is always 4 times as large as h. Hence, 4 is the "constant" in the equation, and A and h are the "variables." Note that there is a definite **relation** between A and h. A is always 4 **times** h.

EXERCISE 50

Determine the **variables** and the **constants** in each of the following examples. Give reasons for each decision that you make.

1. $c = 2\pi R$
2. $d = 40t$
3. $A = \dfrac{bh}{2}$
4. $x = y + 4$
5. $c = 10m + 25$
6. $A = s^2$
7. $P = 2b + 2h$
8. $c = 10m + 50$

GRAPHS SHOW THE **RELATION** BETWEEN TWO VARIABLES

Section 68. Each of the three methods shows relationship. A cost graph, such as Fig. 115, really shows the **relation** between the number of units (lb., doz., or yd., etc.) purchased and the total price paid. A graph of the movement of a train (*e.g.* Fig. 114) which runs at a constant rate shows the **relation** between the number of hours (the time) and the number of miles traveled (the distance). Saying that these graphs show the **relation** between the numbers represented by them means that if we read a particular value of the time, such as 2 hr. or 5 hr., we can find the number of miles which *corresponds* to that number of hours. Thus, graphs show the **relation** between two variables; that is, they show the **values** of one variable **which correspond** respectively to the values of another **related variable.**

A formula also shows the relation or connection between the two variables. For example, the formula for the area of any rectangle with a 3-inch base, which is $A = 3\ h$, shows that *the value of A must always be three times the value of h*, or, in other words, the area is always three times the height. At first it is more difficult to understand the formula than the graph, but as you advance in mathematics the formula will become more important and significant.

Thus, as we stated at the beginning of the chapter, there are three methods of showing the relationship or connection between the kinds of variables we have studied:

I. THE TABULAR METHOD
II. THE GRAPHIC METHOD
III. THE FORMULA METHOD

Let us illustrate for one example each of these methods.

A man walks at the rate of 6 miles per hour. Show the relation or connection between the distance he walks and the number of hours he walks.

I. Tabular method

Table 14

If the no. of hours is	1	2	5	8	10	12
then the distance is	6	12	30	48	60	72

The table shows the relationship between the time spent and the distance traveled.

II. Graphic method

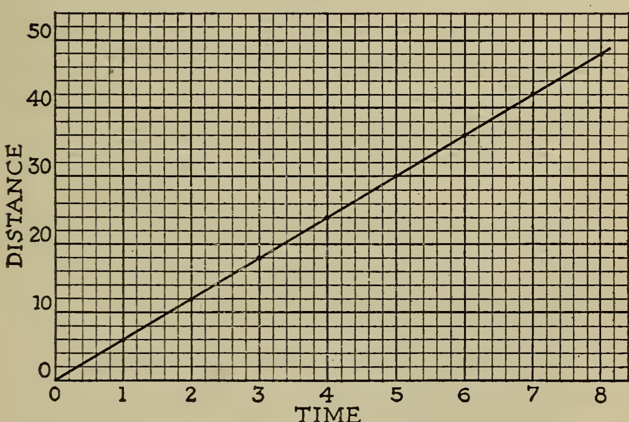


Fig. 116. **The line shows the relationship between the time spent and the distance traveled.**

III. Formula method

$$D = 6\,h.$$

The equation shows the relationship between the time spent and the distance traveled.

EXERCISE 51

PRACTICE IN REPRESENTING THE RELATION BETWEEN VARIABLES

Show by three methods the relation between the variables in the following:

1. The area of a rectangle whose base is 8 in. and its height.
2. The cost of belonging to a club which charges an initiation fee of 50¢, and 10¢ per month for dues.
3. A freight train leaves Chicago at 10 A.M., at the rate of 25 miles per hour; at 1 P.M. a passenger train leaves Chicago, running in the same direction, at the rate of 40 miles per hour. Show graphically at what time the passenger train will overtake the freight train. See Fig. 117 for solution. How does the graph show that one train will overtake the other? If t represents the time of the freight train, what formula will rep-

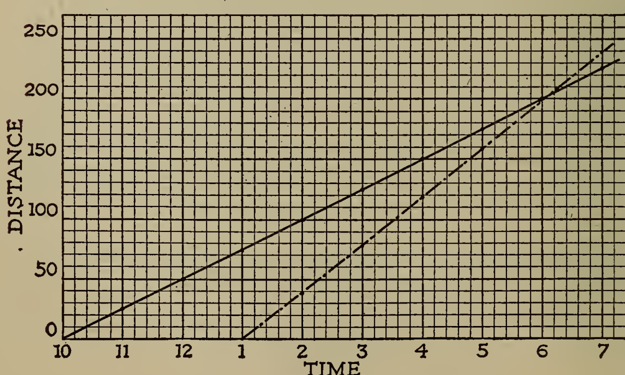


FIG. 117. **The lines show relationship between the time spent and the distance traveled by each train. The point of intersection indicates the time at which they will meet and how far each travels.**

resent the distance it travels? What will represent the time the passenger train travels? What formula will represent its distance?

4. A slow train left Cleveland at 6 A.M., running uniformly at the rate of 30 miles per hour. At 10 A.M. a faster train left Cleveland, running in the same direction, at the rate of 40 miles per hour. Show graphically at what time the faster train will overtake the slower one.

5. A freight train left St. Louis at 7 P.M., running 30 miles per hour. At 11.30 P.M. an express train started in the same direction. Show graphically at what time it will overtake the freight train, if it runs 45 miles per hour.

6. A bicyclist left a certain place at 10 A.M., traveling at the rate of 8 miles per hour. How long would it take a second bicylist to overtake the first one, if he travels at the rate of 10 miles per hour, and starts 2 hours later?

7. An elementary school graduate began work with a weekly wage of \$ 9, and received an increase of 25 ¢ every week. One high school graduate began work at a weekly wage of only \$ 6, but received an increase of 40 ¢ every week. Make a graph which shows the wage of each at the beginning of every week for 20 weeks. From the graph tell when they will be receiving the same wage.

Section 69. Two different kinds of data which can be graphed. We should distinguish between the **two kinds of data which we have graphed.**

A. **The statistical graph. The first kind** includes all those for which no formula or equation can be made. Recall the first illustrative example in this chapter: *the relation between the time of day and the temperature.* Clearly, no formula can be made which will always show the relation between the two variables in this kind of example. Thus, there are only two ways to show or represent this kind of relation: (1) **the tabular method,** (2) **the graphic method.**

B. **The mathematical law. The second kind** of example which we have been graphing is illustrated by any of those examples *for which we made a formula*. For example, we have such illustrations as: the *graph showing the relation between the distance traveled* by a train running at 30 miles per hour *and the time the train travels.* This belongs to the second kind of graph, because we can make a formula for the *relation* between its variables. The formula is:

$$d = 30\,t.$$

Thus, there are **three ways** to show the relation between such variables as these: (1) the **formula** or *algebraic* method, (2) the **tabular** method, and (3) the **graphic** method.

In mathematics, we say that the second kind of graph, for which an equation can always be made, states **algebraic laws, or mathematical laws, because there is always a definite relation between the variables.** The first kind of graph, for which no definite *law* or *equation* can be made, is sometimes called a **statistical graph.** It is this kind that is most frequently seen in newspapers and magazines. In mathematics, however, the other kind, that which states "laws," is nearly always used.

The next exercise will give practice in making both

kinds of graphs. It is important to tell whether the information to be graphed (generally called the *data*) can be expressed by an algebraic law or formula.

EXERCISE 52

1. The following table shows the average heights of boys of different ages. Construct a graph showing this information or data.

Age in years	2	4	6	8	10	12	14	16	18	20
Height in feet	1.6	2.6	3.0	3.5	4.0	4.8	5.2	5.5	5.6	5.7

   Represent ages on the horizontal scale.

2. When does the average boy grow the most rapidly? the most slowly?
3. Is there an algebraic "law," or formula, which shows the relation between these two variables, *age* and *height?*
4. Mr. Smith joined a lodge which charged \$25 initiation fee, and dues of \$2 per month. Show graphically the **relation** between the *cost* of belonging and the *time* one belongs.
5. Is there an algebraic "law," or formula, which shows the **relation** between the variables, *cost* and *time?*
6. The information or data of the following table represent the area of a square of varying sides: Show this **relation** between the area of the square and its side graphically, using the verti-

If the side is	1	2	3	4	5	6	7	8	9	10
then the area is	1	4	9	16	25	36	49	64	81	100

cal scale to measure *areas* and the horizontal scale to measure *sides.*

7. Is there an algebraic "law," or formula, which shows the **relation** between the variables here?

8. The weights of a baby boy who weighed 8 lb. at birth are given for each month of his first year by the table:

Month	1	2	3	4	5	6	7	8	9	10	11	12
Weight	$9\frac{1}{4}$	$11\frac{1}{4}$	$12\frac{3}{4}$	$14\frac{1}{4}$	$15\frac{1}{2}$	$16\frac{1}{2}$	18	19	$19\frac{1}{2}$	20	21	22

Represent this graphically.

9. Can you make a formula to show the relation between weights and months? Then what kind of graph is this, statistical or algebraic?

10. This table shows the cost of various amounts of flour.

If the no. of pounds is	1	2	3	5	10
then the cost is ($)	.08	.16	.24	.40	.80

11. Can you make a formula which shows the relation between these variables? If you should graph these variables, what kind of graph would you get, a straight line, or a broken line?

12. Each member of a class made problems for the others to solve. One boy presented the following table:

If the small no. is	4	7	10	20	50	2
the large no. is	8	14	20	40	100	4

He wanted to see if anybody in the class could make a formula showing the relation between the two numbers. (Use l and s for the numbers.) Could they?

13. The next day a more difficult problem was given. The object was to make a formula to show the relation between the numbers (variables) in the following table:

If the small no. is	2	3	4	5	10
the large no. is	5	7	9	11	21

Can you solve the problem?

14. Make a graph of the formula for example 13.

15. Make up a problem similar to examples 13 and 14 for the members of your class to solve.

EXERCISE 53

PRACTICE IN SOLVING EASY EQUATIONS

Find the value of the unknown in each of the following: Check each example.

1. $2x - 3 = 14$

2. $12 + x = 3x$

3. $\frac{1}{2}y - 2 = 18$

4. $3b + 2b = 2b + 21$

5. $5c - 2 = 13$

6. $120 = 4x + 20$

7. $8a - 9 = 91$

8. $6x = x + 28$

9. $10y - 9 = 47$

10. $12b = 30 + 7b$

11. $\frac{2}{3}y = 18$

12. $\frac{4}{5}x + 2 = 10$

13. $y + \frac{2}{3}y + 5 = 30$

14. $\frac{2}{3}b + \frac{1}{5}b = 26$

15. $\frac{x}{4} = \frac{x}{5} + 1$

16. $\frac{2}{5}a = \frac{3}{10}a + 2$

17. $\frac{5}{4}y + \frac{2}{7}y = y + 15$

18. $20 = \frac{3}{2}y$

SUMMARY OF CHAPTER VIII

This chapter should make clear the following truths:

1. **Important facts about quantities, which change together, are more easily read and interpreted if they are represented *graphically*.**

2. **Graphs always show the relation between two varying quantities.**

3. **There are three fundamental methods of describing the relationship between related variables:**
   a. **The *Graphic* Method of expressing "Law";**
   b. **The *Tabular* Method of expressing "Law";**
   c. **The *Formula*, or Algebraic Method, of stating "Law."**

   They tell the same thing, the graphic method most clearly.

4. **We deal with two important kinds of quantities: (1) Variables, quantities which are continually varying; (2) constants, quantities which are fixed or unvarying.**

5. **It is important to distinguish between two kinds of graphs: (1) Statistical graphs for which no law or relationship can be expressed; and (2) graphs of mathematical laws.**

EXERCISE 54

1. If n represents a boy's present age, state in words what the expression $n + 7 = 22$ means.

2. Give a formula for the base of a rectangle when the area and height are known.

3. Represent the number of cubic yards in a box-shaped excavation when the dimensions are expressed in feet.

4. If m, s, and d represent the minuend, subtrahend, and difference respectively, what formula will show the relation between these numbers?

5. Show by a formula the relation between the product, p, multiplicand, M, and multiplier, m.

6. Give the meaning of the formula $i = prt$.

7. Divide each side of the formula $V = lwh$ by lw and tell what the resulting formula means.

8. Give a formula for the volume of a cube whose edge is s.

9. Evaluate the above formula when $s = 3.2$.

10. Translate into words the formula $d = rt$.

11. Divide each side of the formula $d = rt$ by r and tell what the resulting formula *means*.

12. In the formula $c = np$, n represents the number of articles bought, p represents the price of each, and c represents the total cost. Translate it into a word statement.

13. Divide each side of the formula $c = np$ by n, and tell what the resulting formula means.

14. Does $x = 4$ satisfy the equation $x^2 + 6x = 40$?

15. Solve the equation $10y + 7 = 52 + 4y$.

16. Is the equation $x + y + 3 = 20$ satisfied if $x = 8$ and $y = 9$? Can you find any other values of x and y which will satisfy this equation?

17. Solve each of the following equations, thinking of each example as asking a question:

(*a*) $\frac{10}{x} = 2$ (*e*) $\frac{22}{y} = 5.5$

(*b*) $.5\,y = 17$ (*f*) $1.5\,b = 45$

(*c*) $.4\,p = 80$ (*g*) $\frac{x}{.42} = 60$

(*d*) $\frac{34}{x} = 4.25$ (*h*) $\frac{h}{.34} = 85$

Tell what you do to each side of the equation; that is, tell whether you add, subtract, multiply, or divide, on each side.

REVIEW EXERCISE 53

1. Write in algebraic language: The volume of a sphere is four thirds the cube of the radius times π.

2. Which would you rather have, $4\,x + 5\,y$ dollars, or $5\,x + 4\,y$ dollars, if $x = \$\,20$ and $y = \$\,16$?

3. If V is the volume of a cone, b the area of its base, and h its height, then

$$V = \tfrac{1}{3}\,bh.$$

Write this formula in words.

4. Nurses keep temperature records of fever patients. For one patient the following degrees of fever were noted:

2, 5.4, 4, 6.1, 4.5, 6.5, 5.3, 6.7, 4.5, 6.5, 5.9, 6.2, 6.9, 5, 6.4, 4.7, 5.8, 7.6.

These readings were taken an hour apart. Show graphically this patient's successive temperatures.

5. If you know that

$$10\,a - 7 = 4\,a + 35,$$

then what do you do to each side to get

$$10\,a = 4\,a + 42\,?$$

How do you get $6\,a = 42$?

Why does $a = 7$?

Prove that $a = 7$.

6. Get the hourly temperature for 24 hours from your daily paper, and construct a graph to represent the changes in temperature.

7. The sum of two numbers is 18 and their difference is 4. What are the numbers?

8. Show that $2\,a$ and a^2 are unequal by choosing some particular value for a, such as $a = 6$. Do you think there is any possible value for a which would make $2\,a = a^2$?

9. What product is obtained by using 7 as a factor twice? by using $2\,a$ as a factor three times?

10. The following table shows the number of feet required to stop an automobile running at various speeds.

At a speed of (miles per hour)	10	15	20	25	30	35	40	50
a car should stop (feet)	9.2	20.8	37.	58.	83.3	104.	148.	231.

Represent this graphically, measuring the speed on the horizontal axis.

CHAPTER IX

THE USE OF POSITIVE AND NEGATIVE NUMBERS

Section 70. We need numbers to represent opposite qualities, or numbers of opposite nature. The examples in the following exercises will illustrate what is meant by **opposite qualities,** or numbers of **opposite nature.** We shall take four different kinds of illustrations: (1) **opposite** numbers on a temperature scale, (2) **opposite** numbers on a distance scale, (3) **opposite** numbers to represent financial situations ("having" and "owing"), (4) **opposite** numbers on a *time* scale, to represent "time before" a beginning point and "time after."

FIRST ILLUSTRATION: OPPOSITE NUMBERS ON A TEMPERATURE SCALE

EXERCISE 56

1. The top of the mercury column of a thermometer stands at zero degrees (0°). During the next hour it *rises* 3°, and the next it *rises* 4°. What is the temperature at the end of the second hour?
2. The top of the mercury column stands at 0°. During the next hour it *falls* 3°, and in the next it *falls* 4°. What is the reading at the end of the second hour?
3. If it starts at 0°, *rises* 3°, and then *falls* 4°, what is the reading?
4. If it starts at 0°, *falls* 3°, and then *rises* 4°, what is the reading?

These examples show that we must distinguish two kinds of temperature readings, (1) those *above* zero and (2) those *below* zero. People have agreed to call readings above zero **POSITIVE**, and readings below zero **NEGATIVE.**

Thus, if the mercury starts at zero and rises 4°, it will be at positive 4°, or, more briefly, $+4°$. But if it starts at zero and falls 4°, it will be at negative 4°, or $-4°$. In the remainder of these examples you should describe the mercury readings as *positive* or *negative*, rather than as *above* or *below* zero.

5. The temperature stands at zero. Its first change is described by the expression $+6°$. Its next change is described by $+4°$. What is the temperature at the end of the second change?

6. If the temperature reading is 0°, and it makes the change $-5°$, then $-3°$, what is the final reading?

SECOND ILLUSTRATION: OPPOSITE NUMBERS ON A DISTANCE SCALE

EXERCISE 57

1. An autoist starts from a certain point and goes east 10 miles, and then east 8 miles. How far and in what direction is he from the starting point?

2. If he had first gone west 10 miles, and then west 8 miles, how far and in what direction would he have been from his starting point?

3. If he had first gone east 10 miles and then west 8 miles, how far and in what direction would he have been from his starting point?

4. If he had first gone west 10 miles, and then east 8 miles, how far and in what direction would he have been from his starting point?

These examples show that we must distinguish between **opposite** distances, those *east* of some starting point, and those *west* of the starting point. People have agreed to call distances *east* of the starting point **positive** and distances *west* of the starting point **negative.** By this means a great deal of time can be saved, **because a positive or negative number tells both the *direction* and the *distance*** of a point on the distance scale, from some beginning point. Thus, on the distance scale, Fig. 118, point A is *completely described* by the number -5.

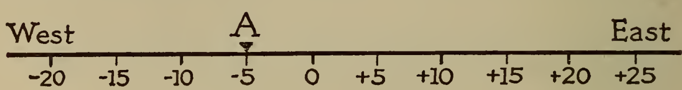


FIG. 118. **Points on a distance scale.**

This number, -5, tells that the point A is 5 units *west* of, or to the left of, the starting point.

5. What would be the position on this distance scale of a man who starts at the zero point, goes east 60 units, and then west 15 units?
6. Where would you be if you started at zero, went $+8$ units, and then -8 units?
7. A man starts at 0; at the end of the first day he is at $+20$, and at the end of the second day he is at -10. What is the total distance he traveled? What number will completely describe his position at the end of the second day?
8. How far is it from $+9$ to -6? What direction is it?

Section 71. Thus, positive and negative numbers are used to distinguish between opposite qualities. The foregoing examples show that we need a brief, economical way

to denote opposite qualities of numbers. This is done by *positive* and *negative* numbers, or, as we shall say from now on, by **SIGNED NUMBERS**. Thus, in referring to temperature readings, *e.g.* the "signed" number, + 10°, shows (1) how far and (2) in what direction the mercury stands from the zero point. In describing the location of a point on a distance scale, the "signed" number, − 6, tells how far and in what direction the point is from the starting or zero point; that is, 6 units to the left of, or to the west of, the zero point.

THIRD ILLUSTRATION: OPPOSITE NUMBERS USED TO REPRESENT FINANCIAL SITUATIONS

Section 72. Positive and negative numbers, or SIGNED NUMBERS, are used also to describe financial situations. It has been agreed to consider money that you "have" as **positive** and money that you "owe" as **negative**. Thus, if you *owe* 40 cents (*i.e.* − 40 cents) and *have* 55 cents (+ 55 cents), your *real* financial situation is + 15 cents. Why? Or, if you *owe* 90 cents (− 90 cents) and *have* 75 cents (+ 75 cents), your *real* financial situation is − 15 cents.

FOURTH ILLUSTRATION: OPPOSITE NUMBERS ON A TIME SCALE

Signed numbers are used also to distinguish "time before" from "time after" a given time. For example, if time before Christ is *negative*, then time after Christ is *positive*. Thus, on the *time scale* below, since Christ's birth is regarded as zero, if a man was born 10 years be-

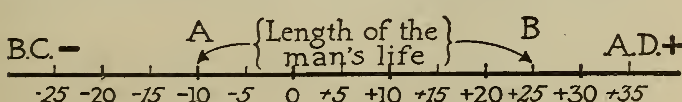


FIG. 119. **Points on a time scale.**

fore Christ and lived 35 years, *the distance* between the points A and B would represent the period of his life. Why?

OTHER ILLUSTRATIONS OF THE USE OF SIGNED NUMBERS: FOR THE PUPIL TO DEVELOP

EXERCISE 58

1. Show how signed numbers are helpful in dealing with latitude; with longitude. Illustrate each one.
2. Show that signed numbers are a convenience in keeping *scores* in games in which you either make or lose a certain number of points.
3. Can you think of any other illustrations of *opposite* numbers?

EXERCISE 59

PRACTICE IN USING SIGNED NUMBERS

1. Your teacher's financial situation is $-\$250$. What does this mean?
2. A man's property is worth \$5200 and his debts amount to \$3300. How can positive and negative numbers be used to represent these amounts? What number will describe his *net* financial situation?
3. The mercury at 8 A.M. was at $-6°$. If it was rising $3°$ per hour, where was it at 9 A.M.? at 10 A.M.? at 11.30 A.M.?
4. Show on a time scale that Cæsar began to rule the Roman people 31 years B.C., and ruled for 45 years.

5. What is the total number of miles traveled by a man who starts at zero on the distance scale if he is at $+6$ at the end of the first day, -2 at the end of the second day, and at -8 at the end of the third day?

6. On the *distance scale*, where would you be if you started at -4 and went east 6 miles?

7. If your financial condition is $+60$ cents, -15 cents, and -12 cents, what single number will accurately describe your net financial situation?

8. What was your final score in a game in which you made the following single scores: $+15$, -8, -10, $+14$, and $+15$?

9. Represent on a *distance scale* (horizontal) the point where a man would be at the end of the third day if he started at zero and walked $+6$ miles on Monday, -10 on Tuesday, and -3 on Wednesday.

10. Find the net financial situation of a man who is worth the following: (*a*) $+\$5+\$8+\$10-\$6$; (*b*) $+6d-10d-8d+15d$.

Section 73. Absolute value of positive and negative numbers. The numerical value of a positive or negative number, without regard to its sign, is its *absolute value*. For example, the absolute value of $+6$ is 6; of $+17$ is 17; of -9 is 9, etc.

Section 74. We need to be able to add, subtract, multiply, or divide signed numbers. Now that we see clearly the practical ways in which positive and negative numbers are used we need to be able to solve problems which contain either kind. In all the examples which we have worked

previously, **only positive numbers have been used.** Next, therefore, we must learn (1) how to *combine* signed numbers (*i.e.* add them); (2) how to *multiply* them; (3) how to find the *difference* between two signed numbers; and (4) how to *divide* signed numbers. We will take them up in that order.

I. HOW TO COMBINE SIGNED NUMBERS: FINDING ALGEBRAIC SUMS

Section 75. When the numbers are arranged vertically. In the example: "Find the net financial situation of a man who is worth the following: $+\$5$, $+\$8$, $-\$10$, $-\$6$," we found *one* signed number which described the man's net financial situation; namely, $-\$3$. That is, we found one signed number which was the result of putting several signed numbers together. This process is called **combining signed numbers**, or **finding the algebraic sum.** Thus, to combine $+4$, -2, -6, and $+3$, we must find one signed number which is the result of putting all of these together. Evidently, this must be -1. Similarly, combining, or finding the algebraic sum of $+5d$ and $-11d$, we get $-6d$.

In each of the following examples, find the algebraic sum, *i.e.* find one number which will describe the result of putting all the separate numbers together. **When no sign is given it is regarded as positive.**

EXERCISE 60

1. $\begin{array}{r} +\$6 \\ -\$3 \\ \underline{+\$4} \end{array}$ **2.** $\begin{array}{r} -7d \\ +8d \\ \underline{-4d} \end{array}$ **3.** $\begin{array}{r} +4x \\ -3x \\ \underline{-\ \ x} \end{array}$ **4.** $\begin{array}{r} 5a \\ 4a \\ \underline{-6a} \end{array}$

5. $$\begin{array}{r} 6y \\ -5y \\ -\ y \\ \hline \end{array}$$

6. $$\begin{array}{r} +8 \\ -5 \\ -6 \\ +7 \\ \hline \end{array}$$

7. $$\begin{array}{r} -10 \\ +13 \\ -\ \ 8 \\ +\ \ 5 \\ \hline \end{array}$$

8. $$\begin{array}{r} +3b \\ -5b \\ -6b \\ +2b \\ \hline \end{array}$$

9. $$\begin{array}{r} 4a \\ -6a \\ \hline \end{array}$$

10. $$\begin{array}{r} -3x \\ -5x \\ +8x \\ \hline \end{array}$$

11. $$\begin{array}{r} 15ab \\ 12ab \\ \hline \end{array}$$

12. $$\begin{array}{r} -\ \ 7xy \\ +11xy \\ -\ \ 2xy \\ \hline \end{array}$$

13. $$\begin{array}{r} +2\frac{1}{2}x \\ -7\ x \\ +2\ x \\ \hline \end{array}$$

14. $$\begin{array}{r} -\ \ 4 \\ +\ \ 3 \\ +11 \\ -\ \ 8 \\ +\ \ 1 \\ \hline \end{array}$$

15. $$\begin{array}{r} x \\ 2x \\ -6x \\ \hline \end{array}$$

16. $$\begin{array}{r} 3a \\ -4a \\ 7a \\ -9a \\ \hline \end{array}$$

Section 76. When the numbers are arranged horizontally. The numbers to be combined are almost always written in a horizontal line, rather than in a vertical column. Combining these terms is done in the same way as if they were written in a vertical column. For example, $+6-5+4-9=-4$.

17. $6-8-5+11=?$

18. $-8-9+11+6$

19. $-6d+5d+9d-2d$

20. $5abc+6abc-7abc$

21. $4f-3f+6f-9f$

22. $+8t-9t-6t+12t$

23. How have you found the algebraic sum of these numbers?

24. $4x-5x-2x+x-10x$

25. $-3ab+5ab-8ab+ab$

26. $2t + 5t - 9t - t$

27. $a^2 + 4a^2 - 7a^2 - 9a^2 + 2a^2$

28. $-y + 12y - 15y + y - 8y$

29. $5x - x - 2x - 3x - 4x - 10x$

30. $8 - 5 - 9 - 2 - 7 + 10$

31. $s^3 + 6s^3 + 2s^3 - 5s^3 - 12s^3$

32. $abc - 7abc - 9abc - 3abc$

Section 77. Terms are either LIKE or UNLIKE: How to distinguish TERMS. Any algebraic expression, such as $ax + b$ or $x^2 + 2xy + 5$, is made up of one or more numbers separated from each other by + signs or by − signs. These numbers thus separated from each other are called **terms**. Thus, in $ax + b$ there are *two* terms; ax is one, b is the other; while in $x^2 + 2xy - 5$ there are three terms, x^2, $2xy$, and 5. Note carefully that a "*term*" *includes everything* between + or − signs.

In many algebraic expressions these terms are all **like** terms, and, as we learned in the previous section, can be combined or put together into one number or term which we called their *algebraic sum*. Thus, $4d$, $+5d$, $-8d$, $+3d$ are **similar** or **like terms**, and their algebraic sum is $+4d$. It is important to understand that because each letter in the expression represents the same thing these are **like** terms. In many cases, however, the terms of an algebraic expression are not all *like terms*.

For example, consider: 4 boys, + 5 girls, + 8 boys, − 2 girls, or, using the initial letters of the words, $4b + 5g + 8b - 2g$. Evidently, these are **unlike terms** and *cannot* be combined into *one* number. However, the *like* terms in the expression *can* be combined; that is, the $+4b$ and $+8b$, giving $12b$, and the $+5g$ and $-2g$, giving

$+ 3g$. Thus the expression $4b + 5g + 8b - 2g$ can be *simplified* or *expressed more briefly* by combining **like terms**, giving $12b + 3g$. From this illustration we see that the **like terms** of any algebraic expression can be combined, giving a simpler, briefer expression than the original one.

EXERCISE 61

FURTHER PRACTICE IN COMBINING SIGNED NUMBERS: NUMBERS HAVING LIKE OR UNLIKE TERMS

Write in the simplest or briefest form each of the following expressions:

1. $2a + 3a - 6a + 4a$
2. 5 ft. + 6 in. − 2 ft. − 4 in.
3. 7 yr. + 3 mo. − 2 yr. − 1 mo.
4. $4b + 5c - 8b - 2c + b$
5. $6a^2 + 3a^2 - 7a^2 + 4a^2$
6. $-2x^3 - 5x^3 - 8x^3 + 2x^3$
7. $ax + 5 + 4ax + 3$
8. $3xy + 5ab - 7xy - 11ab$
9. $2b^3 - 7b^3 + 5b^3 - b^3$
10. $2r + 8r + 3r - 10r$
11. $4a^2b + 5a^2b - 8a^2b - 3a^2b$
12. $-6 + 4 - 8 + 6 - 9 + 2$
13. $5x + 3 - 8x - 4 + 4x + 1$
14. $a^2b + 4a^2b - 6a^2b$
15. $xy + 3 - 8xy - 9 + 2xy + 7$
16. $p + 2q - 8p + 4q + 6p + 5q$
17. $3x + 5y - 7x - 8y - x + y$

18. $5\,ab + 6\,xy - 6\,ab - 7\,xy + 2\,ab + 2\,xy$
19. $-3\,b^2 + 4\,a^2 - 7\,b^2 - 9\,a^2 + 12\,b^2 - 3\,a^2$
20. $8\,y^3 - t - 11\,y^3 + 5\,t + 3\,y^3 - 9\,t$
21. $4 + 6\,x - 9 - 11\,x + 2 + 7\,x - 1$
22. $a^3 + b^3 - 2\,a^3 - 3\,b^3 + 5\,a^3 - 2\,b^3$
23. $2\,t - x - t + 5\,x + 7\,t - 10$
24. $5\,y^2 - 4 + 11\,y^2 + 7 + 6\,y^2 - 3$
25. $7\,abc^2 - 9\,a^2bc - 11\,abc^2 + 11\,a^2bc + abc^2$

SUMMARY OF IMPORTANT PRINCIPLES CONCERNING THE COMBINING OF SIGNED NUMBERS

Section 78. You have now worked many examples in finding algebraic sums. From your experience with such examples, complete these three sentences which tell how to combine signed numbers:

1. **To find the algebraic sum of two positive numbers, __?__ the absolute values of the numbers, and give to the result a __?__ sign.**
2. **To find the algebraic sum of two negative numbers, __?__ the absolute values of the numbers, and give to the result a __?__ sign.**
3. **To find the algebraic sum of two numbers which have unlike signs, find the __?__ of their absolute values, and give to the result the sign of the number which has the __?__ absolute value.**

PRACTICE EXERCISE B: COLLECTING TERMS

Practice on this exercise until you can reach the standard, 10 examples right in 3 minutes. Record the number which you try and the number which you do right. Compare your record each time with the standard.

1.	$2x^2+3-5x^2-7+4x^2-6$ . . . . .
2.	$-2b+c+5b-4c-2c-6b$ . . .
3.	$8a^3-2b+7a^3+3b-15a^3-b$ . .
4.	$5-3x^2+4-x^2+2x^4-8$ . . . .
5.	$ac+6-3ac-13-2ac+5$ . . . . .
6.	$2y^4-5x-8y^4-4x+6y^4+9x$ . .
7.	$2b^2-3b-5b^2-7b+6b^2+8b$ . .
8.	$4p-9-7p+6-8p-13$ . . . . . .
9.	$7-8s^2-11+5s^2+4+3s^2$ . . . . .
10.	$-ax-5ax+b-2b+4ax+5b$.
11.	$3s-5t^5+8s-14s+6t^5-11t^5$.
12.	$-12-8w-4-9w+16+17w$.
13.	$-3e^3+4-e^3+1+5e^3-7$ . . . . .
14.	$2x+3y-11x-13y+5x-7y$ . .

Section 79. Equations solved by addition of signed numbers. What you have just learned about signed numbers will now be used in solving equations. For instance, in solving the equation

$$n-10=-6$$

it is necessary to add $+10$ to each side of the equation. This gives the equation

$$n=+4$$

Again, in solving a difficult equation such as,

$$5y-24=-2y-3$$

it is necessary to add $+24$ to each side. (Why?) This gives the equation

$$5y=-2y+21$$

Then, $-2y$ must be added to each side. (Why?) This gives the equation

$$7y = 21.$$

EXERCISE 62. Oral Work

1. $x - 8 = +10$
2. $y - 2 = +2$
3. $5 = x - 2$
4. $p - 7 = -2$
5. $t - 6 = -4$
6. $x - 8 = -10$
7. $x - 10 = -12$
8. $y - 6 = -8$
9. $p - 4 = -4$
10. $t - 7 = -10$
11. $b - 7 = +8$
12. $2x = 15 - x$
13. $3y = 12 - y$
14. $2b = 20 - 2b$
15. $b - 10 = -2$
16. $5y = 16 - 3y$
17. $4q = -q + 15$
18. $b = -2b + 12$
19. $y = -y + 15$
20. $2c = 10 - 3c$
21. $10 + x = -10$
22. $8 + 2y = +12y$
23. $5y - 2 = 18 - y$
24. $8p - 5 = 35 - 2p$
25. $7r - 8 = -r + 24$
26. $4b - 1 = -2b + 17$
27. $12t - 100 = -13t + 25$
28. $4x - 50 = 70 - x$
29. $7x - 5x - 8x + 14x = 30$
30. $-5y - 4y - y + 12y = 18$

II. HOW TO MULTIPLY SIGNED NUMBERS

Section 80. The four ways to multiply signed numbers. In arithmetic it was found that multiplication shortened the work of addition. For example, in adding $3 + 3 + 3 + 3 + 3 + 3 + 3$, the result is found most easily by multiplying 3 by 7, because 3 is taken 7 times. So, in algebra, it is equally desirable to multiply one signed number by another.

There are *four* different ways in which we may have to multiply signed numbers. These are:

(1) **plus** times **plus**, as in the example $+4$ times $+2 = ?$

(2) **plus** times **minus**, as in the example $+4$ times $-2 = ?$

(3) **minus** times **plus**, as in the example -4 times $+2 = ?$

(4) **minus** times **minus**, as in the example -4 times $-2 = ?$

By considering the following problems we can tell what meaning must be given to the multiplication of signed numbers.

A. ILLUSTRATIVE QUESTIONS BASED UPON THE SAVING AND WASTING OF MONEY

EXERCISE 63

1. If you save \$5 a month ($+$\$5), how much better off will you be 6 months from now ($+6$)?
Evidently you will be \$30 better off ($+$\$30).
Thus, $+5$ times $+6 = +30$.

2. If you have been saving \$5 a month ($+$\$5), how much better off were you 6 months ago (-6)?
Evidently you were \$30 worse off ($-$\$30) than you are now. Thus, $+5$ times $-6 = -30$.

3. If you are wasting \$5 a month ($-$\$5), how much better off will you be in 6 months from now ($+6$)?
Evidently you will be \$30 worse off ($-$\$30).
Thus, -5 times $+6 = -30$.

4. If you have been wasting \$5 a month ($-$\$5),

how much better off were you 6 months ago (-6)?
Evidently you were \$30 better off $(+\$30)$. Thus, -5 times $-6=+30$.

Summarizing: These problems based upon saving and wasting money have led to the following illustrative statements:

1. $+5$ times $+6=+30$.
2. $+5$ times $-6=-30$.
3. -5 times $+6=-30$.
4. -5 times $-6=+30$.

B. ILLUSTRATIVE QUESTIONS BASED UPON THERMOMETER READINGS

EXERCISE 64

1. If the mercury is now at zero and *is rising* 2° per hour $(+2)$, where *will it be* 4 hours from now $(+4)$?
   Evidently it will be 8° above zero $(+8)$. Thus, $+2$ times $+4=+8$.
2. If the mercury *has been rising* 2° per hour $(+2°)$ and is now at zero, where *was it* 4 hours ago (-4)?
   Evidently it was 8° below zero $(-8°)$. Thus, $+2$ times $-4=-8$.
3. If the mercury is now at zero and *is falling* 2° per hour $(-2°)$, where *will it be* 4 hours from now $(+4)$?
   Evidently it will be 8° below zero $(-8°)$. Thus, -2 times $+4=-8$.

4. If the mercury is now at zero and *has been falling* 2° per hour ($-2°$), where *was it* 4 hours ago (-4)?
Evidently it was 8° above zero ($+8°$). Thus, -2 times $-4 = +8$.

Summarizing: these problems based upon the thermometer have led to the following illustrative statements:

1. $+2$ times $+4 = +8$.
2. $+2$ times $-4 = -8$.
3. -2 times $+4 = -8$.
4. -2 times $-4 = +8$.

A careful study of these illustrations will enable you to complete the following statements concerning multiplication of signed numbers:

1. A positive number multiplied by a positive number gives as a product a __?__ number.
2. A positive number multiplied by a negative number gives as a product a __?__ number.
3. A negative number multiplied by a positive number gives as a product a __?__ number.
4. A negative number multiplied by a negative number gives as a product a __?__ number.
5. **The product of two numbers which have like signs is __?__.**
6. **The product of two numbers which have unlike signs is __?__.**

The last two statements are generally used as **rules for multiplication.** These rules or general statements are based upon the previous illustrations. You should refer to them when in doubt about how to multiply any two signed numbers.

11*a*. PARENTHESES ARE USED TO INDICATE MULTIPLICATION

Section 81. Multiplication of two or more numbers is often indicated by placing the numbers within parentheses. Thus, "$+4$ times -35" is often written "$(+4)(-35)$." It is important to note that *no* sign or symbol is placed between the parentheses when multiplication is indicated.

EXERCISE 65

PRACTICE IN MULTIPLYING SIGNED NUMBERS

1. $(+3)(+5)$
2. $(+6)(-2)$
3. $(+10)(-2\frac{1}{2})$
4. $(+6)(-9)$
5. $(-2)(-5)$
6. $(+8)(-\frac{1}{4})$
7. $(+12)(+6)$
8. $(+5)(+4)(-2)$
9. $(-3)(-6)(+2)$
10. $(-4)(-10)(-3)$
11. $(+\frac{2}{3})(-\frac{6}{8})$
12. $(-\frac{1}{2})(+\frac{5}{8})$
13. $(-7)(-6)(+2)$
14. $(-2)(-2)(-2)$
15. $(-3)(-3)(-3)$
16. $(+1)(+1)(+1)(+1)$
17. $(-2)(-2)$
18. $(-2)(-2)(-2)(-2)$
19. $(3)(4)(5)(2)$
20. $(+2)(+\$5)$
21. $(+3)(+7D)$
22. $(+6)(+5 \text{ ft.})$
23. $(-8)(+6y)$
24. $(+\frac{2}{3})(18a)$
25. $(-12)(+\frac{1}{2})$
26. $(\frac{2}{3})(-18)$
27. $(27)(-\frac{2}{9})$
28. $(+\frac{6}{5})(-25)$
29. $(-32)(-\frac{5}{8})$
30. $(-\frac{3}{4})(+24d)$
31. $(-\frac{2}{3})(+\frac{9}{8})$
32. $(-\frac{5}{7})(-\frac{14}{15})$
33. $(-2)(-2)(+2)$
34. $(+\frac{4}{5})(-\frac{10}{12})$
35. $(\frac{1}{5})(-10)$
36. $-5 \cdot 8$

37. $-7 \cdot 21$

38. $(-6)(-\frac{1}{6})$

39. $2 \cdot 2 \cdot (-3)$

40. $(-1)(-1)(-1)(-1)$

41. $(2)(-3)(+4)(-5)$

42. $(6)(\frac{1}{2})(\frac{1}{3})$

43. $(10)(5)(-\frac{1}{20})$

44. $(\frac{1}{2})(-\frac{4}{3})$

II *b*. HOW TO USE EXPONENTS IN MULTIPLICATION

Section 82. Suppose we had to find the product of $3x^2$ and $5x^4$. It is important to keep in mind the meaning of exponents. $3x^2$ means $3 \cdot x \cdot x$ and $5x^4$ means $5 \cdot x \cdot x \cdot x \cdot x$. Hence, $3x^2$ times $5x^4$ or $(3x^2)(5x^4)$ means $3 \cdot x \cdot x \cdot 5 \cdot x \cdot x \cdot x \cdot x$, or $15x^6$. By the same reasoning, the product of $+6x^5$ and $-7x^4$ is $-42x^9$.

IMPORTANT PRINCIPLE OF USING EXPONENTS IN MULTIPLICATION

From such examples as these we can state a very important principle:

> **The exponent of any letter in the product is equal to the sum of the exponents of that letter in the separate factors.**

EXERCISE 66

PRACTICE IN USING EXPONENTS IN MULTIPLYING SIGNED NUMBERS (ORAL)

1. $(5a^2)(6a^4)$

2. $(+7b)(-9b^5)$

3. $(+8)(2y^3)$

4. $(6ab)(2a^2)$

5. $(+3abc)(5ab)$

6. $4x^3 \cdot 5x^2 \cdot 2x^5$

7. $y^4 \cdot 5y^3$

8. $\frac{1}{2}x^3 \cdot 10x^7$

9. $2ab^2 \cdot 3ab$

10. $a^2b \cdot ab^2 \cdot a^3b^4$

11. $x \cdot 2x^3$

12. $-6a \cdot 3a$

13. $-2y \cdot 3y^2$

14. $(-5b)(-2b^2)$

15. $(-6x^2)(-7x^2y)$

16. $(-8x^2)(\frac{1}{4}x)$

17. $(+10y^2x)-(2yx^2)$
18. $16a^2b\cdot(\frac{1}{4}ab^3)$
19. $a\cdot b\cdot b\cdot a\cdot a$
20. $x^4\cdot 2x$
21. $y\cdot 5y^3$
22. $b^2c\cdot bc$
23. $5\cdot x^2$
24. $10y^2\cdot\frac{1}{10}y\cdot y^3$
25. $x^2\cdot 3x\cdot x^3$
26. $-2y\cdot y^2$
27. $(-x)(-5)(x^2)$
28. $(-y^2)(+y^5)(-y^3)$
29. $(\frac{1}{2}t^4)(-6t)(\frac{2}{3})$
30. $(-\frac{3}{4}a)(-\frac{4}{3}a^5)$
31. $(-\frac{10}{2}y)(-1)(-1)$
32. $(ab^2)(-a^2bc)$
33. $(+8y^3x)(-\frac{1}{2}x^3y^2)$
34. $(-10a)(\frac{1}{10}a^2)$
35. $(-x^2)(-x^3)(-y^2)$
36. $(-y^5)(+y^3)(-y^2)$
37. $(-\frac{3}{5}y^3)(-\frac{4}{3}x)(y)$

Write a rule for the use of exponents in multiplication.

REVIEW EXERCISE 67

Perform the indicated operations in the following examples:

1. $5a^2+2b-8a^2-b+a^2$.
2. $(4x^3)(2x)(-3x^2)(-x^5)$.
3. Represent, by a drawing, two squares, each side of the first being $2a$, and each side of the second $4a$.

   (*a*) What is the perimeter of each square? What is the ratios of their perimeters?

   (*b*) What is the area of each square? What is the ratio of their areas?
4. A final examination contained the following question: Give both the algebraic sum, and the product, of the following expressions:

(*a*) $-8, +2, -1, +10, -\frac{1}{2}$.

(*b*) $4x, 5x^2, 2x, -3x^2$.

(*c*) $y^2, 2y^3, -5y^2, -7y^3$.

Why do you think this was a difficult question?

5. What is the shortest way to write the sum of five x^2's? the product of five x^2's?

6. Evaluate a^3 when $a = -1$; when $a = -2$; when $a = -5$.

7. Determine the numerical value of $a^2b - ab^2$ when $a = 5$ and $b = 2$.

8. $x^2 + x^3 + 2x^2 + 2x^3 + 3x^2$.

9. $x^2 \cdot x^3 \cdot 2x^2 \cdot 2x^3 \cdot 3x^2$.

10. What algebraic expression will represent the area of Fig. 120? Evaluate the expression when $x = 5$ and $y = 4.5$.

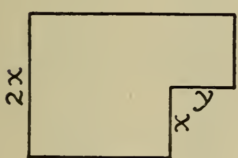


FIG. 120

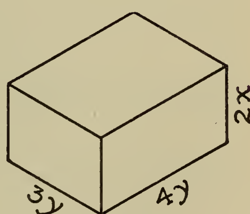


FIG. 121

11. Find the area of all the faces or sides of the rectangular box, Fig. 121. What expression represents the volume? What would the volume be if each dimension were $4y$?

12. Add: $4x - 3y + 2$, $5x + 6y - 8$, $-7x + 3y - 9$.

III. HOW TO FIND THE DIFFERENCE BETWEEN SIGNED NUMBERS: SUBTRACTION

Section 83. How "differences" are found in practical work. Clerks in stores have a method of making change or of finding the difference between two numbers that is very helpful in finding the difference between two *signed* numbers. For example, if a customer gives the clerk 50 cents in payment for a 27-cent purchase, the clerk begins at 27 and counts out enough money to make 50 cents. If we use the same terms as were used in arithmetic, — namely, the *subtrahend*, *minuend*, and *difference*, — then we say, "The clerk begins at the subtrahend, 27 cents, and counts to the minuend, 50 cents."

First illustrative example. To illustrate this method of finding the difference between two signed numbers, let us consider this problem:

On a certain day the mercury stands at $-4°$ in Chicago and at $+13°$ in St. Louis. How much warmer is it in St. Louis, or what is the difference between $+13°$ and $-4°$? Naturally, we do the same thing the clerk does, *begin at the subtrahend and count to the minuend*, *i.e.* we count from $-4°$ to $+13°$, giving us $+17°$. The difference is called *positive* because we counted *upward.* If we counted *downward*, the difference would be called *negative.* This example is written as follows:

$$\begin{array}{rl} +13° & \text{minuend} \\ \underline{-\ \ 4°} & \text{subtrahend} \\ +17° & \text{difference} \end{array}$$

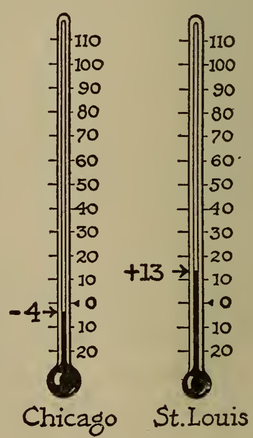


FIG. 122

Second illustrative example. Subtract $+10$ from -5 by referring to the number scale. This means to find the distance from the subtrahend to the minuend or from $+10$ to -5. The distance from 10 above to 5 below is clearly 15; and since the direction is downward, the *difference* is -15. This example is written:

-5	minuend
$+10$	subtrahend
-15	difference

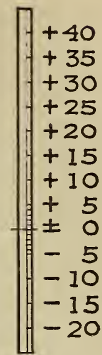


Fig. 123

These illustrations are given merely to show that **the difference between two signed numbers can always be found by counting on a number scale from the subtrahend to the minuend.** The difference will be positive or negative, depending upon whether the direction of counting is *upward* or *downward.*

EXERCISE 68

PRACTICE IN FINDING THE DIFFERENCE BETWEEN TWO SIGNED NUMBERS: SUBTRACTION

1.	$+6$	-4	$+8$	$+3$	$+5$	-7	$+13$	$+9$
	-2	$+5$	$+2$	$+10$	-8	$+4$	$+4$	$+14$

2.	$+\$4$	$-6°$	$+7d$	$+10$ ft.	-4 in.	$+3x$
	$-\$8$	$+9°$	$-2d$	-6 ft.	-7 in.	$-10x$
3.	$-3a$	$+4b$	$-5c$	$10x^2$	$-3x^3y$	$+10abc$
	$-11a$	$-2b$	$+6c$	$-2x^2$	$+5x^3y$	$-4abc$

4.	$-2t^3$	$+5xy^2$	$5a$	$3x$	$-2bc$
	$+3t^3$	$+11xy^2$	$-2a$	x	bc

5. From $-7a$ take $+5a$
6. Take $-13b^2$ from $+2b^2$
7. From x take $7x$
8. Take $-bc$ from $2bc$
9. $(6)-(-2)$
10. $(+4)-(+4)$
11. $(-2)-(+8)$
12. $(-3y)-(+2y)$
13. $(8x^2)-(10x^2)$
14. Take -6 from -4
15. From $+2$ take -15
16. From $+12y^2$ take $-8y^2$
17. Diminish $+6$ by -4
18. $(+2)+(-8)-(-6)$
19. $(-2)-(+6)-(-10)$
20. Find the value of $a-b$, if $a=10$, $b=-5$.

Section 84. Subtraction without the use of a scale. In arithmetic it is often stated that subtraction is the process of finding what number must be added to one number, the subtrahend, to produce another number, the minuend. For instance, in subtracting 12 from 20, you are to think what number must be "**put with**" or "**added to**" 12 to give 20. Obviously 8 is the number. Let us apply the same thinking to signed numbers.

Illustration. Find the difference between + 12 and − 4. Think what must be "put with" or "added to" −4 to give +12. Clearly it is + 16

$$\begin{array}{r} +12 \\ \underline{-\ 4} \\ +16 \end{array}$$

Thus you see **another method of subtracting signed numbers,** *i.e.* you **find out what number must be "put with" or "added to" the subtrahend to produce the minuend.**

EXERCISE 69

Solve these examples by thinking as you did in the explanation given above.

1. $\begin{array}{r} +10 \\ -\ 2 \\ \hline \end{array}$

2. $\begin{array}{r} +\ 8 \\ -\ 6 \\ \hline \end{array}$

3. $\begin{array}{r} +12 \\ -10 \\ \hline \end{array}$

4. $\begin{array}{r} +\ 2 \\ -18 \\ \hline \end{array}$

5. $\begin{array}{r} +12 \\ +15 \\ \hline \end{array}$

6. $\begin{array}{r} +\ 4 \\ +\ 9 \\ \hline \end{array}$

7. $\begin{array}{r} +\ 5 \\ +11 \\ \hline \end{array}$

8. $\begin{array}{r} +\ 2 \\ +\ 8 \\ \hline \end{array}$

9. $\begin{array}{r} -\ 8 \\ -\ 3 \\ \hline \end{array}$

10. $\begin{array}{r} -12 \\ -\ 7 \\ \hline \end{array}$

11. $\begin{array}{r} -25\,t \\ -18\,t \\ \hline \end{array}$

12. $\begin{array}{r} -11\,x \\ -\ 7\,x \\ \hline \end{array}$

13. $\begin{array}{r} -\ 3 \\ -\ 9 \\ \hline \end{array}$

14. $\begin{array}{r} -\ 5y \\ -11y \\ \hline \end{array}$

15. $\begin{array}{r} -\ 8\,x^2 \\ -12\,x^2 \\ \hline \end{array}$

16. $\begin{array}{r} -\ 7\,xy \\ -12\,xy \\ \hline \end{array}$

17. $\begin{array}{r} +2\,x+3\,y \\ -5\,x-8\,y \\ \hline \end{array}$

18. $\begin{array}{r} -3\,t+5\,w \\ +8\,t-2\,w \\ \hline \end{array}$

19. $\begin{array}{r} 2\,x-y \\ -7\,x+5\,y \\ \hline \end{array}$

20. $(12)-(-3)$

21. $(-15)-(+12)$

22. $(-14)-(+14)$

23. $5\,x-(-4\,x)$

24. $1-(-6)$

25. $x-(5\,x)$

26. $4\,y-(-3\,y)$

Section 85. Solving equations which necessitate the subtraction of signed numbers. In Chapter II you solved equations by subtracting certain numbers from each side. For example, the equation $3\,x+10=22$ is solved by subtracting 10 from each side. In such examples the subtraction did not require the use of signed numbers. However, if we wish to solve an equation such as

$$3\,x+10=-8,$$

we must subtract $+10$ from each side of the equation. That requires us to subtract signed numbers; that is, we subtract $+10$ from -8, which gives us -18. Then we have the equation

$$3\,x=-18$$

or

$$x=-6$$

EXERCISE 70

PRACTICE IN SOLVING EQUATIONS WHICH INVOLVE SUBTRACTION

1. $x + 8 = 10$
2. $2y + 3 = 11$
3. $5c + 1 = 21$
4. $6a + 2 = 20$
5. $x + 12 = 10$
6. $y + 6 = 4$
7. $b + 11 = 7$
8. $c + 2 = 2$
9. $y + 5 = -1$
10. $b + 4 = -8$
11. $c + 10 = -4$
12. $y + 15 = -3$
13. $b + 4 = -4$
14. $t + 9 = -9$
15. $p + 10 = 10$
16. $11 + b = 8$
17. $2y + 10 = 8$
18. $9b + 12 = 3$
19. $4 + 6x = 2$
20. $8 + y = -4$
21. $6 + 4x = -6$
22. $10 + p = -2$
23. $12 + 5y = -13$
24. $16 + 8x = -16$
25. $2p + 20 = -4$
26. $6b + 4 = 2b - 8$
27. $10 + 8p = -4 + p$
28. $2x - 10 = 12$
29. $3y - 8 = -20$
30. $10c - 2 = c + 16$

IV. HOW TO DIVIDE SIGNED NUMBERS

Section 86. Division is the opposite of multiplication. You will have little or no difficulty in the **division** of signed numbers if you understand that division is just the opposite of multiplication. For example, if $4 \times 2 = 8$, then $\frac{8}{2} = 4$. In this case 8 is the dividend, 2 is the divisor, and 4 is the quotient. In signed numbers, as well as in arithmetic, the *dividend equals* the *quotient times* the *divisor.*

$$+8 \div -2 = -4;\text{ or } \frac{+8}{-2} = -4,\text{ because } (-2)(-4) = +8.$$

$-8 \div -2 = +4$; or $\frac{-8}{-2} = +4$, because $(-2)(+4) = -8$.

$+8 \div +2 = +4$; or $\frac{+8}{+2} = +4$, because $(+2)(+4) = +8$.

$-8 \div +2 = -4$; or $\frac{-8}{+2} = -4$, because $(+2)(-4) = -8$.

EXERCISE 71

PRACTICE IN FINDING THE QUOTIENTS OF SIGNED NUMBERS

Find the quotient in each of the following:

1. $\frac{+10}{+2}$; $\frac{+18}{-2}$; $\frac{-16}{+4}$; $\frac{-30}{-10}$; $\frac{-14}{+7}$; $\frac{+6}{+2}$; $\frac{-8}{-1}$; $\frac{+16}{-2}$.

2. $\frac{+15\,d}{-3}$; $\frac{-18 \text{ ft.}}{+3}$; $\frac{+16 \text{ mo.}}{-8}$; $\frac{+25\,x}{-5}$; $\frac{-10\,a}{+2}$.

3. $(21)\div(-3)$; $(-36)\div(-9)$; $(-54)\div(+27)$; $(-96)\div(+12)$; $(-21)\div(-7)$; $(-60)\div(+12)$.

4. $\frac{10\,x^3}{2\,x}$; $\frac{21\,y^4}{3\,y^2}$; $\frac{18\,b^5}{6\,b^4}$; $\frac{12\,c^4}{6\,c^4}$; $\frac{16\,p^3}{16\,p}$. How can you prove each of these?

5. $\frac{+20\,y^2}{-4\,y}$; $\frac{-27\,x^6}{-9\,x^4}$; $\frac{-34\,b^2c^2}{-17\,b^2}$; $\frac{-50\,x^3y}{+25\,y}$; $\frac{-72\,y^6w^5}{-6\,yw^5}$.

6. What is a good way to **check** or prove your work in division?

EXERCISE 72

PRACTICE IN DIVISION WHICH INVOLVES SIGNED NUMBERS

1. Divide each of the following by $2\,a$: $4\,a$, $6\,a^2$, $-8\,a$, $-10\,a^2$, a, $-12\,a^3$, $+12\,a^5$, $-6\,a^2x$, $-11\,a^3y$.

2. Divide each of the following by $-5y$: $10y$, $-5y^2$, $-15y^3$, y^2, $-y^3$, $10xy$, $-25x^2y^3$, $-xy$.

3. If $-3y = 24$, what does $+y$ or y equal? If $-5x = 30$, what does $+x$, or x, equal?

4. Solve each of the following equations:

(*a*) $-2x = 12$	(*h*) $-5x + 2 = -8$
(*b*) $-5y = 20$	(*i*) $-4b - 3 = -19$
(*c*) $4y = 12$	(*j*) $7 - 8y = -49$
(*d*) $-6b = -18$	(*k*) $2 - c = -5$
(*e*) $-9c = 90$	(*l*) $5 + 3y = -4$
(*f*) $-c = 10$	(*m*) $14 - 5b = -6$
(*g*) $-2x = -14$	(*n*) $5 - y = -4$

EXERCISE 73

COMPLETING STATEMENTS ABOUT DIVISION

1. A positive number divided by a positive number gives as a quotient a ___?___ number.

2. A negative number divided by a positive number gives as a quotient a ___?___ number.

3. The quotient of two numbers having like signs is ___?___.

4. The quotient of two numbers having unlike signs is ___?___.

5. The exponent of any letter in the quotient is equal to the exponent in dividend ___?___ the exponent in the divisor.

EXERCISE 74

A REVIEW OF ADDITION, MULTIPLICATION, SUBTRACTION, AND DIVISION OF SIGNED NUMBERS

This is a very important exercise.

1. From the sum of $2a$ and $-5a$ take the difference between $-3a$ and $+8a$.
2. Add the product of -3 and $+5$ to the quotient of -18 and -2.
3. Take the sum of $-7b$ and $+4b$ from the difference between $-6b$ and $+11b$.
4. To the quotient of -21 and $+3$ add the product of -6 and $+7$.
5. From the sum of $7t$ and $-10t$ take the difference between $-4t$ and $+11t$.
6. Add the product of -6 and $+9$ to the quotient of -28 and -4.
7. Take the sum of $-9x$ and $+3x$ from the difference between $-5x$ and $+13x$.
8. To the product of -7 and $+11$ add the quotient of -33 and $+3$.
9. From the sum of $8c$ and $-14c$ take the difference between $-5c$ and $+16c$.
10. Add the product of -12 and $+5$ to the quotient of -32 and -4.
11. Take the sum of $-8y$ and $3y$ from the difference between $-7y$ and $+12y$.
12. To the product of -8 and $+9$ add the quotient of -36 and $+6$.
13. From the sum of $12b$ and $-16b$ take the difference between $-7b$ and $+8b$.

14. Add the product of -8 and $+9$ to the quotient of -40 and -5.

15. Take the sum of $-11\,t$ and $7\,t$ from the difference between $-9\,t$ and $+10\,t$.

16. To the product of -6 and $+13$ add the quotient of -42 and $+7$.

SUMMARY OF CHAPTER IX

This chapter shows the following important points:

1. **The need for skill in using signed numbers to represent opposite qualities, such as thermometer readings, distances, etc.**

2. **That like terms represent the same thing in an expression and can be combined; unlike terms represent different things and cannot be combined in one number.**

3. **To find the algebraic sum of two numbers which have unlike signs, find the difference of their absolute values, and give to the result the sign of the number which has the greater absolute value.**

4. **You reviewed the principle that whatever is done to one side of an equation must also be done to the other side. For example, in your practice with equations you added to, subtracted from, or divided by the same signed number on each side of the equation.**

5. **The product of two numbers which have like signs is positive; the product of two numbers which have unlike signs is negative.**

6. **The exponent of any letter in a product is equal to the sum of the exponents of that letter in the separate factors.**

7. **The quotient of two numbers which have like signs is positive; the quotient of two numbers which have unlike signs is negative.**

REVIEW EXERCISE 75

1. The formula $h = 25 + \frac{3}{2}(G - 4)$ is used to determine the proper height of the chalk trough in a schoolroom. If h stands for the height in inches, and g stands for the number of the grade, find the height for Grade VIII; that is, when $g = 8$. What is the proper height for a third-grade room?

2. Evaluate the expression $ab^2 + a^2b$ if $a = 2$ and $b = -3$.

3. Show that the sum of any two numbers having unlike signs, but the same absolute value, is zero. Give some illustrations.

4. In a class of 25 pupils, 2 were conditioned and 6 failed. Express the ratio of the number of pupils that succeeded to the total number in the class. What percentage is this?

5 The ratio of $y + 1$ to 9 is equal to the ratio of $y + 5$ to 15. Find y.

The number of posts required for a fence is 84 when they are placed 18 feet apart. How many would be needed if they were placed 12 feet apart?

6. If I am now x years old, what does the following expression tell about my age: $2x + 5 = 55$?

CHAPTER X

THE FURTHER USE OF THE SIMPLE EQUATION

A. HOW TO SOLVE SIMPLE EQUATIONS WHICH CONTAIN NEGATIVE NUMBERS AND PARENTHESES

Section 87. What we have already learned about the equation. Since the equation is a very important part of mathematics, we must be able to solve quickly and accurately equations of any kind. Thus far we have learned two very important facts about equations:

1. That if we do anything to one side of an equation, we must **do the same thing to the other side.**

2. That an equation is **solved** when a value of the unknown is found which **satisfies** the equation; that is, one which makes the numerical value of one side equal to the numerical value of the other side, when the value is substituted for the unknown.

Furthermore, you have learned: (1) how to solve simple equations of the type

$$6b + 3 = 45,$$

or, $$c + 5c = 20 + c, \text{ etc.};$$

(2) how to get rid of fractions in an equation, *e.g.* of the type $\frac{2}{3}x + \frac{3}{4}x - 1 = 3$;

(3) how to solve word problems, first by translating them into equations and second by solving the equations. These methods, which you have now mastered, are important first steps in the more important problem of learning how to solve equations of **any kind**.

I. SOLVING EQUATIONS WHICH CONTAIN NEGATIVE NUMBERS

Section 88. There are just **two more steps** that we must learn in using equations. **First,** we must be able to **solve**

equations which contain negative numbers; **second**, we must be able to **solve equations which contain parentheses.** Negative numbers occur very commonly in equations. The following examples illustrate this fact.

EXERCISE 76

Write as equations, and solve each of the following examples:

1. What number multiplied by 7 equals -28?
2. What number multiplied by -5 equals 20?
3. If a certain number be added to 13, the result is 8. Find the number.
4. A certain number increased by 10 equals -5. Find the number.
5. If 7 be subtracted from a certain number, the result is -3. What is the number?
6. If negative four times a certain number gives 22, what is the number?

These examples show how negative numbers occur in equations. Throughout the remainder of the work, equations which are satisfied by negative numbers will occur very commonly. The next exercise contains many examples of this kind.

EXERCISE 77

SOLUTION OF EASY EQUATIONS WHICH CONTAIN NEGATIVE NUMBERS

Solve each of the following equations. You should be able to tell exactly what must be done to each side of the equation.

1. $x+5=3$	3. $b+7=2$	5. $2x+16=2$
2. $2y=-16$	4. $-3a=15$	6. $-4y=12$

7. $10y + 2 = -18$

8. $2b - 1 = 9$

9. Three times a certain number, increased by 10, gives 6. What is the number?

10. If twice a certain number is added to 16, the result equals the number increased by 6. Find the number.

11. $\frac{2x}{3} + 5 = \frac{1}{4}x + \frac{1}{6}x + 2.$

12. $12 - 2x = 8.$

13. The sum of two thirds of a certain number and three fourths of the same number is -17. Find the number.

A new kind of equation

14. $-2x - 12 = 5x - 40.$

The equations which you have just solved are of the kind in which you can easily see what to do to each side. With examples like 14, however, in which both *knowns* and *unknowns* occur on *each* side and which *include negative numbers* on one or both sides, we need special and systematic practice.

Section 89. We need to get "knowns" on one side and "unknowns" on the other. Just as clerks in stores always place the known weights on one scale pan and the unknown weights on the other scale pan, so we, in solving equations, always get the **known numbers**, or terms, **on one side of the equation**, and the **unknown terms on the other side.**

Usually we get all the **unknown** terms on the **left** side, and all the **known** terms on the **right** side. Thus, in the equation above,

$$-2x - 12 = 5x - 40,$$

we do not want -12 on the left side. Therefore we get rid of the **known** on the left side by adding $+12$ to **each** side, giving the equation $-2x = 5x - 28$. We also do

not want the $5x$ on the right side. Therefore we subtract $5x$ from **each** side, giving the equation $-7x = -28$. Dividing **each** side of this equation by -7, we find that $x = 4$.

Section 90. Equations should be solved in a systematic order. In learning to solve equations which require several steps, **pupils make many mistakes** because their work is not arranged in a set order. For the present, therefore, you will find it very important to use a form like the following:

Illustrative example. Solve the equation

$$-2x - 12 = 5x - 40.$$

(1) Adding $+12$ to each side gives

$$-2x = 5x - 28.$$

(2) Subtracting $5x$ from each side gives

$$-7x = -28.$$

(3) Dividing each side by -7 gives

$$x = 4.$$

(4) Check: Substituting 4 for x gives

$$-8 - 12 = 20 - 40.$$

$$-20 = -20.$$

EXERCISE 78

Solve each of the following examples, **writing out each step** exactly as in the solution of the illustrative example:

1. $-3x - 8 = 8x - 30$
2. $5y - 6 = 9y + 42$
3. $-6b + 11 = 2b + 43$
4. $x - 20 = 50 - 6x$
5. $-2c + 10 = 4$
6. $10 - 3x = -20$
7. $6 - 4y = 2$
8. $14 = 2y + 20$
9. $-7a + 4 = +8a - 41$
10. $-2x - 7 = -8x - 19$
11. $+5y - 3 = 8y - 16$
12. $5 + 2y = 0$
13. $10x + 22 = 12$
14. $0 = 4x + 20$

Section 91. A short way to solve an equation: To TRANSPOSE terms from one side of the equation to the other. In solving the preceding equations you were often

forced to add or subtract something on each side of the equation. For example, see the illustrative example on page 199. This becomes very laborious. The work may be reduced very much by using the following short method. Several examples will illustrate it clearly.

The old way of thinking about the equation.	The short cut for doing it.
1. $x - 7 = 5.$ Add 7 to each side, gives $x - 7 + 7 = 5 + 7,$ or $x = 12.$	1. $x - 7 = 5.$ Transpose the -7, gives $x = 5 + 7,$ or $x = 12.$
2. $4x + 9 = 1.$ Subtract 9 from each side, gives $4x + 9 - 9 = 1 - 9,$ or $4x = -8.$	2. $4x + 9 = 1.$ Transpose the $+9$, gives $4x = 1 - 9,$ or $4x = -8.$
3. $-2x - 12 = 5x - 40.$ Add 12 to each side, gives $-2x - 12 + 12 = 5x - 40 + 12,$ or $-2x = 5x - 28,$ and then Subtract $5x$ from each side, giving $-7x = -28.$	3. $-2x - 12 = 5x - 40.$ Transpose -12, and $+5x$, gives $-2x - 5x = -40 + 12.$ Collecting terms, $-7x = -28.$

In the *first* example, we wanted to get rid of -7 on the left side of the equation. By the short method, the -7 is put on the other side as $+7$.

In the *second* example, we wanted to get rid of $+9$ on the left side. The short method simply puts -9 on the other side.

In the *third* example, we wanted to get rid of -12 on the left side. The short way of doing it is to put $+12$ on the other side. We also wanted to get rid of $5x$ on the right side; the short way of doing it is to put $-5x$ on the other side.

These examples show that **any term** in an equation may be **transposed** from one side of the equation to the other side, **by changing its sign.** Transposition is merely a device for shortening the labor required in *adding* or *subtracting* the same number on both sides of the equation.

The student should solve the following example by the **old method,** to see which he likes the better.

Solve :	$-3x-8=8x-30.$
Transposing,	$-3x-8x=-30+8.$
Collecting,	$-11x=-22.$
Dividing by -11,	$x=2.$
Check :	$-6-8=16-30.$
	$-14=-14.$

EXERCISE 79

PRACTICE IN SOLVING EQUATIONS BY TRANSPOSING TERMS

Solve the first three equations by both methods:

1. $7x+34=-4x-10$
2. $8y-11=-2y+9$
3. $6c+13=-17+c$
4. $12x+8=-32+2x$
5. $0=27-6x-3$
6. $4y-9y=-2y-39$
7. $6a-11a+40=0$
8. $13p-27=20p+8$
9. $23x-82=30x+2$
10. $-12y=-18-6y$
11. $17-3b=-8b-8$
12. $1-9p=9p+10$
13. $0=2x-12-14$
14. $4c=16+6c$
15. $4x-7=53-6x$
16. $\frac{1}{3}x+\frac{1}{6}=\frac{1}{2}x$
17. $\frac{3}{4}y+5=20$
18. $\frac{x}{9}-\frac{1}{3}=-2$
19. $\frac{p+4}{5}=p$
20. $\frac{2x-8}{5}=\frac{4-x}{6}$
21. $\frac{2n+3}{5}+n-\frac{3}{2}$
22. $-8x-17=2x-47$
23. $0=5y-18+y$

II. HOW TO SOLVE EQUATIONS CONTAINING PARENTHESES

Section 92. We saw in the last chapter that **parentheses** were used to indicate multiplication. Thus, to show that -4 is to be multiplied by -6, we use the **parentheses**, as follows: $(-4)(-6)$. Multiplication is usually indicated in this way. Take this example to illustrate the way in which parentheses will be used in equations:

Illustrative example.

Solve the equation which states that the perimeter of the rectangle in Fig. 124 is 54 inches.

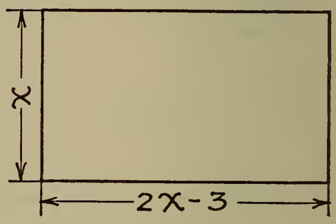


FIG. 124

Solution: $2 \cdot x =$ length of the two altitudes.

$2(2x-3) =$ length of the two bases.

Therefore, $2x + 2(2x-3) = 54$.

Note the use of parentheses, *i.e.* to show that the expression $2x-3$ must be multiplied by 2.

(1) Removing parentheses, gives

$$2x + 4x - 6 = 54.$$

(2) Transposing, $\quad 2x + 4x = 54 + 6.$

(3) Collecting, $\quad 6x = 60.$

(4) Whence $\quad x = 10$, the altitude

and $\quad 2x - 3 = 17$, the base.

EXERCISE 80

PRACTICE IN SOLVING EQUATIONS WHICH CONTAIN PARENTHESES

Solve and check each of the following equations:

1. $2(x+10)=42$
2. $5(y-2)=15$
3. $3(2b-4)=18$
4. $4x+5(x+2)=46$

5. $2(x-3)+3(x-2)=8$

6. $5b+2(4-b)=32$

7. $-3x+6(x-4)=9$

8. $-7b+4(2b-3)=16$

Section 93. Note that in all the foregoing examples the number before the parenthesis has been *positive*. **If negative numbers occur, however, we proceed just the same,** remembering how to multiply a negative number.

Illustrative example.

Solution of an equation involving REMOVAL OF PARENTHESES.

9. $8x-2(2x-7)=x+8.$

The expression $2x-7$ is to be multiplied by -2.

(1) Performing this multiplication, or *removing parentheses*, gives

$$8x-4x+14=x+8. \quad \text{(Why is it } +14\text{?)}$$

(2) Transposing, gives

$$8x-4x-x=8-14.$$

(3) Collecting, $3x=-6.$

(4) Dividing each side by 3 gives

$$x=-2.$$

(5) Check: Substituting -2 for x throughout the equation gives

$$-16-2(-4-7)=-2+8.$$
$$-16+8+14=-2+8.$$
$$6=6.$$

EXERCISE 80 (*continued*)

10. $5b-3(4-2b)=2b+42$

11. $6(x-3)-4+(x\ 2)=4-x$

12. $7(b-2)-2(3+b)=0$

13. $4(2y-5)+15=3(y+10)$

14. $9y-3(2y-4)=6$

Section 94. A difficult form of multiplication. A form of multiplication that gives pupils difficulty is the kind represented by $-(4-5x)$ in the equation:

$$15.\quad 5-2(x-6)=-(4-5x)$$

When no multiplier appears immediately before the parentheses, the multiplier 1 is understood. Therefore in this case, it is just as though the right side of the equation read

$$-1(4-5x).$$

Illustrative example.

The complete set of steps required to solve this equation includes:

(1) Removing parentheses gives

$$5-2x+12=-4+5x. \quad \text{(Why is it } +5x\text{?)}$$

(2) Combining terms gives

$$-2x+17=-4+5x.$$

(3) Transposing gives

$$-2x-5x=-4-17.$$

(4) Collecting,

$$-7x=-21.$$

(5) Dividing each side by -7 gives

$$x=3.$$

(6) Substituting 3 for x, throughout the *original* equation to *check* the result, gives

$$5-2(3-6)=-(4-5\cdot 3).$$

or,
$$5-6+12=-4+15.$$
$$11=11.$$

There are two **important and difficult points** in this last example. **First**, you should note that in the expression $5-2(x-6)$ the -2 is NOT to be subtracted from the 5. The expression in parentheses must be **multiplied** by -2. **Second**, if no multiplier is written before the parentheses, as in the expression $-(4-5x)$, it is understood that the multiplier is 1. In this case it is -1. If there had been

no sign before the parentheses, as $(4 - 5x)$, the multiplier would be understood to be $+1$.

EXERCISE 80 (*continued*)

16. $7x-(x-4)=25$

17. $-5y-(2-y)=18$

18. $6x-(x+7)=-2x+35$

19. $5-2(x-4)=23$

20. $7-12(3-b)=31$

21. $2b-7(3-b)=b+8$

22. $1(2x+3)=-17$

23. $-1(6-2x)=43$

24. $-(6-2x)=24$

25. $16=(2x+4)$

A SUMMARY OF THE STEPS REQUIRED TO SOLVE EQUATIONS WHICH CONTAIN PARENTHESES

Section 95. Look back to the illustrative examples, 9 and 15, and compare the steps in the solution that is worked out for each one with the steps in the solution in each of those which you have just worked. You will note that to solve such an equation the following steps are always included:

I. Removing the parentheses (*i.e.* multiplying).

II. Combining like terms on each side.

III. Transposing to get all the knowns on one side and all the unknowns on the other.

IV. Dividing each side by the coefficient of the unknown, to give the numerical value of the unknown.

V. Substituting the obtained value in the original equation to *check* the result.

REVIEW EXERCISE 81

1. $\frac{x}{5}+x=18$

2. $\frac{2}{3}y+\frac{4}{7}y=2$

3. $3(x-5)-x=-23$

4. $x-2(x-4)=0$

5. $y-(10-y)-12=2$

6. $2c=4c-16$

7. $-(x-6)-3x=2(x-12)$

8. $\frac{b}{2} - \frac{3b}{5} = -2.$

9. $\frac{1}{n} = \frac{3}{9}$

10. $\frac{2b}{5} = \frac{8}{3}$

11. $-(20 - 3y) = 1$

12. $.2y - .5y = 120$

13. $.2(6b - 1) = 2.6$

14. $2x - 8x - 20 = -4x + 42$

15. $-14 - 10y = +4y - 84$

STANDARDIZED PRACTICE EXERCISE C (TIMED)

Practice until you can reach the "standard," 12 examples right in 5 minutes.

1. $5 - 2x = 13 - 4x$ . . . .
2. $7b - 4 + b = 3b - 10 + 2$ . . . .
3. $5c + 3 = -2$ . . . .
4. $4a + 8 = 6a - 5$ . . . .
5. $2y - 5 + 8y = 11 + 7y + 2$ . . . . . .
6. $3 = 4x + 9$ . . . .
7. $10 - y = 3y + 18$ . . . .
8. $13 - 5x + 7 = x + 3 - 2x$ . . . .
9. $5b - 7 = 8$ . . . .
10. $-20 - 3c = 7c + 9$ . . . .
11. $11x - 7 - 4x = 5x + 8 - 19$ . . . . .
12. $9 = 4x + 7$ . . . .
13. $8 - 4a = 12 - 6a$ . . . .
14. $9 + x - 6x = 5x + 3 + 21$ . . . .
15. $12b + 3 = -21$ . . . .
16. $21 - 4a = 17 + a$ . . . .
17. $-2x - 8 + 5x = +2x + 7 + 5$ . . .
18. $7x = -12 - x$ . . . .

SECOND TASK OF THE CHAPTER

B. A FURTHER STUDY OF HOW TO SOLVE WORD PROBLEMS ALGEBRAICALLY

Section 96. Review of important steps in translating word problems into algebraic statements. We have taken a great deal of time to learn how to solve any kind of simple equation because we need to be able to use equations skillfully in solving actual problems later. The problems as a rule will not be stated for us, in algebraic or equational form, all ready for solution. They will be stated merely in words. First, then, we shall always have to **translate** the word problem into an equation. After this first step the work is the mere solution of the equation.

Our **second principal task in this chapter**, therefore, is to become skillful in translating word problems into algebraic form. We learned in our work with Chapter II the important steps in translating word problems. Since we are to learn in the next few lessons how to *translate* a great many different kinds of word statements, let us review these steps here:

First step: See clearly which things in the problem are **known** and which are **unknown.**

Second step: Represent one of the unknowns, most conveniently the smallest one, by some letter.

Third step: Represent all of the others by using the same letter.

Fourth step: By careful study of the RELATIONS between the parts of the problem, express the word statement in algebraic form.

Sometimes this will mean an **equation** and sometimes not. For the next few lessons, therefore, you will work many word problems. The exercises are included to give practice in translating many different kinds, **so that you will be able to use the method in solving any kind that you may happen to meet later.** For convenience they will be arranged by types, examples of the same type being studied together.

Section 97. Need for tabulating the data of word problems. Many problems involve so many different statements that it is practically **necessary to arrange the steps in the translation in very systematic tabular form.** Take an example like this:

John's age exceeds James's by 20 years. In 15 years he will be twice as old as James. Find the age of each now.

Before we can write this statement in the form of an equation we must express in algebraic form *four* different things: (1) John's age *now*; (2) James's age *now*; (3) John's age in 15 years; and (4) James's age in 15 years. These four facts can best be stated in a table like this:

(First step) Let n represent James's age NOW.

(Second step) Tabulate the data:

TABLE 15

	Age now	Age in 15 years
John's age	$n + 20$	$n + 20 + 15$
James's age	n	$n + 15$

With all the facts expressed in letters we can now state the equation which **tells the same thing as the original word statement**; namely:

(**Third step**) $n + 20 + 15 = 2(n + 15)$.

We are now ready for the

(**Fourth step**) the solution of the equation; the steps **are** as follows:

(1) $$n + 35 = 2n + 30.$$
(2) $$-n = -5.$$
(3) $$n = 5.$$

Therefore James's age **now** is 5, and John's age **now** is $n + 20$, or 25.

(4) Check the accuracy of this result thus:
In 15 years John will be 40 and James will be 20; or John will be twice as old as James, as the problem states.

To be proficient in solving such problems, therefore, we first need practice in tabulating such facts as "age **now**," "age some other time," as in this example. Other types which involve the same need for tabulation will be taken up later.

I. PROBLEMS RELATING TO AGE

EXERCISE 82

PRACTICE IN REPRESENTING RELATIONS BETWEEN NUMBERS

1. A man is now 25 years of age. What expression will represent his age:

(*a*) 10 years ago? (*c*) x years ago?

(*b*) 8 years from now? (*d*) m years from now?

2. C is now n years of age. What expression will represent his age:

(*a*) 12 years from now? (*c*) y years ago?

(*b*) 7 years ago? (*d*) m years from now?

3. A is now x years old. B's present age exceeds A's age by 8 years. What expression will represent:

   (*a*) B's present age?

   (*b*) the sum of their ages?

   (*c*) the age of each 10 years ago?

   (*d*) the age of each 5 years from now?

   (*e*) the sum of their ages in 5 years?

4. A is now n years of age; B is three times as old. Express algebraically:

   (*a*) B's present age;

   (*b*) the age of each 4 years ago;

   (*c*) the age of each 9 years from now.

   (*d*) State algebraically that B's age 4 years ago was 5 times A's age then.

5. A's present age exceeds B's present age by 25 years. In 15 years he will be twice as old as B. Find their present ages.

6. C is six times as old as D. In 20 years C's age will be only twice D's age 20 years from now. What are their present ages?

7. A man is now 45 years old and his son is 15. In how many years will he be twice as old as his son?

8. A father is 9 times as old as his son. In 9 years he will be only 3 times as old. What is the age of each now?

9. A's present age is twice B's present age; 10 years ago A's age was three times B's age then. Find the age of each now.

II. PROBLEMS IN WHICH A NUMBER IS DIVIDED INTO TWO OR MORE PARTS

Section 98. The solution of a great many problems depends upon our being able to separate a number into two or more parts. For example, if a man has a certain sum of money to invest, he may invest part of it in one thing, and part in another. The solution of such an example requires that we be able to divide a number into two or more parts algebraically.

EXERCISE 83

PRACTICE IN DIVIDING A NUMBER INTO TWO OR MORE PARTS

1. The sum of two numbers is 20.
   (*a*) Express in algebraic form the second one if the first one is 12.
   (*b*) Express in algebraic form the second one if the first one is n.
   (*c*) Express in algebraic form the fact that the second one exceeds the first one by 4.

2. There are 36 pupils in a mathematics class.
   (*a*) Express algebraically the number of boys if there are 19 girls.
   (*b*) Express algebraically the number of girls if there are n boys.
   (*c*) State algebraically that there were 6 more girls than boys.

3. A farmer has two kinds of seed, clover seed and blue grass seed. If he has 100 lb. of both, express:

(*a*) the number of pounds of clover seed if there were 24 lb. of blue grass seed;

(*b*) the number of pounds of clover seed if there were n lb. of blue grass seed;

(*c*) the value of the clover seed (n lb.) at 20 cents per pound; and the value of the blue grass seed at 15 cents per pound.

(*d*) State by an equation that the value of both kinds together was $19.

4. Divide 20 into two parts such that the larger part exceeds the smaller part by 4.

5. A boy paid 48 cents for 20 stamps; some cost two cents each and the remainder cost three cents. How many of each kind did he buy?

6. During one afternoon a clerk at a soda fountain sold 200 drinks, for which he received $16. Some were 5 cents each; the others were 10 cents each. Find the number of each kind.

7. A grocer has two kinds of coffee, some selling at 30 cents per pound and some selling at 50 cents per pound. How many pounds of each kind must he use in a mixture of 100 pounds which he can sell for 34 cents per pound?

III. PROBLEMS BASED ON COINS

Section 99. Another illustrative type of word problem which gives practice in tabulating data and thus in solving difficult word problems is the "coin problem." Take this example:

Illustrative example. A man has 3 times as many dimes as quarters.

How *many* of each has he if the value of both together is \$11?

Here there are four distinct numbers to be expressed, as in the case of the age problem: (1) the *number* of quarters; (2) the *number* of dimes; (3) the *value* of the quarters in terms of a common base (for example, cents); (4) the *value* of the dimes in the same base (cents). The steps in the solution are clear, therefore, from the following illustrative solution:

TABLE 16

	Number	Value (cents)
quarters	n	25 n
dimes	3n	30 n

(1) Let $n =$ the *number* of quarters.

(2) Then $3n =$ the number of dimes.

(3) $25n + 30n = 1100$ cents.

(4) $\therefore n = 20$, number of quarters.

(5) $3n = 60$, number of dimes.

(*a*) *Value* of the quarters = \$5.

(*b*) *Value* of the dimes = \$6.

Total value = \$11, as stated in the example.

EXERCISE 84

PRACTICE IN EXPRESSING THE VALUE OF VARIOUS NUMBERS OF COINS

1. Express the value *in cents* of:

(*a*) d dimes; (*d*) $4d$ half dollars;
(*b*) $3d$ quarters; (*e*) d dollars;
(*c*) $2d$ nickels; (*f*) of all the coins.

2. Express the *value* in cents of:

(*a*) n nickels; (*c*) $(n+5)$ quarters;
(*b*) $(3-n)$ dimes; (*d*) $(12-n)$ half dollars;
(*e*) $(30-n)$ nickels.

3. A purse was found which contained nickels and dimes, 20 in all. Find the number of each if the value of both was $1.60.

4. I received at a candy counter twice as many dimes as quarters, and 6 more nickels than dimes and quarters together. How many of each coin did I receive if the value of all was $7.50?

5. A debt of $72 was paid with 5-dollar bills and 2-dollar bills, there being twice as many of the latter as of the former. Find the number of each kind of bill.

6. 18 coins, dimes and quarters, amount to $2.25. Find the number of each kind of coin.

7. A cab driver received twice as many quarters as half dollars, and three times as many dimes as half dollars; in all he had $13. How many of each coin did he receive?

IV. PROBLEMS BASED ON TIME, RATE, AND DISTANCE

Section 100. In Chapter VIII we saw that the motion of a train could be represented graphically. Now we shall learn how to solve this kind of problem by means of the equation.

EXERCISE 85

PRACTICE IN SOLVING PROBLEMS BASED ON RELATIONS BETWEEN TIME, RATE, AND DISTANCE

1. Express the distance covered by an automobile in 10 hours if its rate is:

   (*a*) 18 miles per hour;
   (*b*) 5 miles per hour;

(*c*) $(r + 3)$ miles per hour;
(*d*) $(2x - 5)$ miles per hour.

2. A train runs for t hours. Express the distance it will cover at the rate of:
   (*a*) 35 miles per hour;
   (*b*) m miles per hour;
   (*c*) $(r + 6)$ miles per hour;
   (*d*) t miles per hour.

3. An automobile tourist sets out on a 400-mile trip. Express the time required if he goes at the rate of:
   (*a*) 40 miles per hour;
   (*b*) 5 miles per hour;
   (*c*) $(r + 10)$ miles per day;
   (*d*) $(2r - 3)$ miles per day.

4. How long will it require to make a trip of D miles at the rate of 15 miles per hour? 5 miles per hour?

5. At what rate must one travel to go D miles in 10 hours? In t hours? In $t + 3$ hours?

6. A slow train travels at the rate of 5 miles per hour; a fast train travels 15 miles more per hour. Express:
   (*a*) the rate of the fast train;
   (*b*) the distance passed over by each in 5 hours.
   (*c*) State algebraically that the two trains together traveled 100 miles in 5 hours.

7. Two trains leave Chicago at the same time, one eastbound, the other westbound. The eastbound train travels 10 miles less per hour than the westbound train. Express:

(*a*) the rate of each;

(*b*) the distance traveled by each in 4 hours.

(*c*) Form an equation stating that they were 440 miles apart at the end of 4 hours.

8. Two trains, 350 miles apart, travel toward each other at the rate of 40 and 35 miles per hour, respectively.

(*a*) Express the distance traveled by each in t hours.

(*b*) Form an equation stating the fact that the trains met in t hours.

9. Make formulas for d, for t, and for r, that can be used in any problem based upon uniform motion.

10. **Illustrative example.** **Two bicyclists, 200 miles apart, travel toward each other at rates of 12 and 8 miles per hour respectively. In how many hours will they meet?**

(1) Let t represent the number of hours until they meet.

(2) TABLE 17

	Time in hours	Rate per hr. in miles	Distance in miles
For slow one	t	8	8t
For fast one	t	12	12t

(3) Then $8t + 12t = 200.$

(4) $\therefore t = 10.$

11. Two men start from the same place, one going south and the other going north. One goes twice as fast as the other. In 5 hours they are 120 miles apart. Find the rate of each.

12. An eastbound train going 30 miles per hour left Chicago 3 hours before a westbound train going 36 miles per hour. In how many hours, after the westbound train left, will they be 519 miles apart?

13. A bicyclist traveling 15 miles per hour was overtaken 8 hours after he started by an automobile which left the same starting point $4\frac{1}{2}$ hours later. Find the rate of the automobile.

14. A starts from a certain place, traveling at the rate of 4 miles per hour. Five hours later B starts from the same place and travels in the same direction at the rate of 6 miles per hour. In how many hours will B overtake A?

REVIEW EXERCISE 86

1. The sum of two numbers is 55; twice the greater equals three times the smaller, plus 15. Find each number.

2. A is 10 years older than C, and B is 6 years younger than C. The sum of their ages six years ago was 40 years. How old is each now?

3. A is now 17 years old and B is 50. In how many years will A be exactly one half as old as B?

4. The value of 36 coins, dimes and quarters, is \$6.60. Find the number of each kind of coin.

5. A collection of nickels, dimes, and quarters amounts to \$4. There are 10 more nickels than dimes, and 2 less quarters than dimes. Find the number of each.

6. One car, running 20 miles per hour, left a certain point 4 hours after another car which was running at the rate of 15 miles per hour. In how many hours will it overtake the other one?

7. A and B start from the same place, and travel in opposite directions. A's rate is twice B's rate. In 4 hours they are 120 miles apart. Find the rate of each. Make a drawing to illustrate.

8. A leaves a certain place at 10 o'clock, traveling due north. B starts from the same place at 12 o'clock, and travels due south. At 4 o'clock they are 210 miles apart. Determine how fast each traveled, if A's rate is one half of B's rate. Make a drawing.

9. A left a town 4 hours after B left. They traveled in opposite directions, A at the rate of 12 miles per hour, and B at the rate of 20 miles per hour. In how many hours will they be 272 miles apart?

10. A freight train left Kansas City for St. Louis at the rate of 12 miles per hour at the same time that a passenger train running 45 miles per hour left St. Louis for Kansas City. How long will it be before they meet, if the distance between the two cities is 285 miles?

11. Two hours after a messenger, who was traveling at the rate of 10 miles per hour, left a camp, it was decided to cancel his message. How fast must a second messenger travel to overtake him in 8 hours?

12. Two autoists 200 miles apart start toward each other. The faster one travels 20 miles per hour and the slower one 15 miles per hour. In how long will they meet, if the faster one is delayed 2 hours on the trip?

V. PROBLEMS INVOLVING PER CENTS

Section 101. Many problems involving per cents may be solved by algebraic methods.

EXERCISE 87

PRACTICE IN SOLVING PERCENTAGE PROBLEMS

1. What does 10% mean? 5%? r%?
2. Indicate 4% of \$600; 5% of \$275.
3. Express decimally 5% of p; 8% of c· $6\frac{1}{2}$% of b.
4. A man paid c dollars for an article. He sold it at a gain of 25%. Express:
   (*a*) the gain in dollars;
   (*b*) the selling price.
   (*c*) State algebraically that he sold the article for \$2.50.
5. A merchant sold a suit for \$25, thereby gaining 25%. If the cost is represented by c dollars, what will represent:
   (*a*) the gain in dollars?
   (*b*) the selling price in terms of c?
   (*c*) State algebraically that the selling price was \$25.
6. Solve each of the following equations:

$$(a)\quad .20x = 180$$
$$(b)\quad x + .06x = 3.18$$

(*c*) $c + .10\,c = 495$
(*d*) $m - .15\,m = 21.25$
(*e*) $p + .04\,p = 520$
(*f*) $x - .50\,x = 18.75$
(*g*) $2 - 3\,x - .5\,x = 7$
(*h*) $1.75\,x - \frac{1}{2}\,x = 1000$

7. Find the cost of an article sold for $156 if the gain was 10 %. (Use *c* for the cost.)

8. What number increased by $66\frac{2}{3}$ % of itself equals 150?

9. After deducting 15 % from the marked price of a table, a dealer sold it for $21.25. What was the marked price?

10. A dealer made a profit of $3690 this year. This is 18 % less than his profit last year. Find his profit last year.

11. A number increased by 12.5 % of itself equals 243. What is the number?

12. A shoe dealer wishes to make 25 % on shoes. At what price must he buy them in order to sell them at $4.50 per pair?

13. A furniture dealer was forced to sell some damaged goods at 14 % less than cost, and sold them for $129. How much did they cost?

14. A man sold a suit of clothes for $30.25. What per cent did he gain if the clothes cost him $25?

VI. INTEREST PROBLEMS

Section 102. Many interest problems can be more easily solved by algebraic equations than by the methods of arithmetic.

EXERCISE 88

1. Express the interest on \$150 at 5 % for 1 year; for 3 years; for t years.

2. Express the interest on P dollars at 6 % for 1 year; for 3 years; for t years.

3. Express the simple interest on \$ 500 for 1 year at r %; for 4 years.

4. A man borrowed a certain sum of money at 6 %. Express:
   (*a*) the interest for 2 years.
   (*b*) State algebraically that the interest for three years was \$ 48.

5. What principal must be invested at 6 % to yield an annual income of \$ 57?

6. For how many years must \$ 2800 be invested at 7 % simple interest to yield \$ 833 interest?

7. What is the interest on P dollars at r % for t years?

8. A man invests part of \$ 1000 at 4 %, and the remainder at 6 %. If x represents the number of dollars invested at 4 %, express:
   (*a*) the annual income on the 4 % investment;
   (*b*) the amount of the 6 % investment;
   (*c*) the annual income on the 6 % investment.
   (*d*) State algebraically that the annual income on the 4 % investment exceeds the annual income on the 6 % investment by \$ 20.

9. Part of \$ 1200 is invested at 5 % and the remainder at 7 %. The total annual income from the two investments is \$ 67. What was the amount of each investment?

10. Ten thousand dollars' worth of Liberty Bonds yield an annual interest of \$370. Some pay $3\frac{1}{2}\%$, and the remainder pay 4%. Find the amount of each kind of bond.

11. A 5% investment yields annually \$5 less than a 4% investment. Find the amount of each investment if the sum of both is \$800.

VII. PROBLEMS CONCERNING PERIMETERS AND AREAS

Section 103. The following examples are based on squares and rectangles:

EXERCISE 89

1. The length of a rectangle exceeds twice its width by 12 in. Represent its width by w.

(*a*) Make a drawing to represent it.
(*b*) Express its length.
(*c*) Express its area.
(*d*) Express its perimeter.
(*e*) State that its perimeter is 84 in. Solve the resulting equation.

2. The length of a rectangle is 9 in. more, and the width is 6 in. less, than the side of a square.

(*a*) Make a drawing for each.
(*b*) Express the dimensions of the square.
(*c*) Express the dimensions of the rectangle.
(*d*) Express the perimeter of the rectangle.
(*e*) Express the area of the rectangle.
(*f*) State algebraically that the sum of the perimeters is 168 in. Solve the resulting equation.

3. The base of a triangle exceeds its height by 10 inches.

(*a*) Make a drawing for the figure.
(*b*) Express its base and height.
(*c*) Express its area.
(*d*) State that its area is equal to the area of a rectangle whose dimensions are 8 in. and 5 in. (Note that you have not yet learned how to solve equations like this. We will take them up in Chapter XVIII.)

4. The length of a rectangle is 4 feet more, and its width is 2 feet less, than the side of a square whose perimeter is P inches. Express:
(*a*) the side of the square;
(*b*) the dimensions of the rectangle;
(*c*) the perimeter of the rectangle.
(*d*) Find the value of P if the perimeter of the rectangle is 44 inches.

5. A tennis court for singles is 3 feet shorter than 3 times its width. Find the length and width of the court, if its perimeter is 210 ft.

6. The width of a basket-ball court is 20 ft. less than its length. The perimeter is 240 ft. Find its dimensions.

7. A picture, twice as long as it is wide, is inclosed by a frame an inch wide. The perimeter of the outer edge of the frame is 44 inches. What is the size of the picture?

8. The length of a rectangle exceeds its width by 10 inches. If each dimension is increased by 5 inches, the resulting perimeter will be 128 inches. Find the area of the original rectangle.

9. If P equals the perimeter of a square, each of whose sides is s, what will be the perimeter of a square each of whose sides is $3s$?

VIII. PROBLEMS BASED ON LEVERS

Section 104. A teeter board is one form of *lever.* The point on which the board rests or turns is the *fulcrum;* the parts of the board to the right of and to the left of the fulcrum are the *lever arms.*

FIG. 125

If a boy at A, who just balances a boy at B, moves to the left while B remains stationary, it is clear that the left side goes down. But if the boy at B moves closer to the fulcrum while A remains stationary, then A goes down. It is also clear that boys of unequal weight cannot teeter unless the heavier boy sits closer to the fulcrum. There is a mathematical relation between the weight on the lever arm and its distance from the fulcrum. Two boys will balance each other when the *weight* of one *times* his *distance* from the fulcrum is equal to the *weight* of the other *times* his *distance*, or in general, when

> ***weight times distance* on one side equals *weight times distance* on the other side.**

This law or relation may be tested by placing equal coins at different positions on a stiff ruler balanced on the edge of a desk. Try this experiment. See whether 2 pennies

placed 4 inches from the fulcrum (at the center of the lever) will balance 1 penny placed 8 inches from the fulcrum. See whether 6 pennies placed 2 inches from the fulcrum will balance 3 pennies placed 4 inches from the fulcrum.

Thus, to make a lever balance, the *product* of *weight* and *distance* from the fulcrum on one side *must equal* the *product* of *weight* and *distance* from the fulcrum on the other side.

EXERCISE 90

PROBLEMS BASED ON LEVERS. MAKE A DRAWING FOR EACH

1. John weighs 80 lb. and sits 4 ft. from the fulcrum. Where must Robert sit if he weighs 90 lb.?
2. A, weighing 120 lb., sits $4\frac{1}{2}$ ft. from the fulcrum, and balances B, who sits 5 ft. from the fulcrum. What is B's weight?
3. A hunter wishes to carry home two pieces of meat, one weighing 40 lb. and the other 60 lb. He puts them on the ends of a stick 4 ft. long and places the stick across his shoulder. Where must the fulcrum (his shoulder) be placed to make the weights balance?
4. Two children play teeter, one on each end of a board 9 ft. long. Where must the fulcrum be if the children weigh 60 and 80 lb. respectively?
5. Could three children teeter on the same board? How?
6. A and B sit on the side of the fulcrum. A weighs 100 lb. and sits 5 ft. from the fulcrum; B weighs 80 lb. and sits 3 ft. from the fulcrum. Where must C sit to balance the other two, if he weighs 150 lb.?

SUMMARY OF CHAPTER X

This chapter has taught all the steps involved in solving a simple equation:

1. **Removal of parentheses.**
2. **Getting rid of fractions.**
3. **Transposing to get all the knowns on one side and all the unknowns on the other.**
4. **Dividing each side by the coefficient of the unknown.**
5. **Checking by substituting the obtained value of the unknown in the original equation.**

Many kinds of word problems have been solved. Tabulating the information or data of such problems is a great help in solving them. A *systematic method* always pays big dividends in any kind of work.

EXERCISE 91

MISCELLANEOUS PROBLEMS

1. A grocer has two kinds of tea, — some worth 60 ¢ per pound and some worth 75 ¢ per pound. He has 20 lb. more of the former than of the latter kind. How many pounds of each kind has he, if the value of both kinds is $45.75?
2. I bought 45 stamps for $1.05. If part of them were 2-cent stamps and part 3-cent stamps, how many of each did I buy?
3. The sum of the third, the fourth, and the eighth parts of a number is 17. What is the number?

4. John has $\frac{1}{3}$ as many marbles as Harry. If John buys 120 and Harry loses 23, John will then have 7 more than Harry. How many has each boy?

5. A clerk spends $\frac{1}{4}$ of his yearly salary for board and room, $\frac{1}{8}$ for clothes, $\frac{1}{6}$ for other expenses, and saves \$880. What are his annual expenses?

6. A father left one third of his property to his wife, one fifth to each of his three children, and the remainder, which was \$1200, to other relatives. Find the value of his estate.

7. Ten years ago A was one third as old as he is at present. Find his age now.

8. Find C in the formula $C = \frac{5(F-32)}{9}$ if $F = 20$.

9. A merchant bought goods for \$500, and sold them at a gain of 5%. What was the selling price?

10. If in problem 9 the merchant had sold the goods at a gain of x per cent, what would have been the selling price?

11. A 6-foot pole casts a shadow $4\frac{1}{2}$ ft. in length. At the same time how long is the shadow of an 8-foot pole?

12. The ratio of two numbers is $\frac{3}{4}$. Find each number if their sum is 56.

13. Two numbers differ by 70; the ratio of the larger to the smaller is $\frac{7}{2}$. Find each number.

14. In Fig. 126, $\angle C = 90°$, $\angle A = 37°$, and $AC = 24$. Find AB, BC, and $\angle B$.

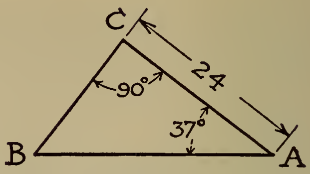


FIG. 126

15. In general, which is the larger, the *cosine* or the *tangent* of an angle? Show by a drawing.

16. The highest office building in the world (the Woolworth Building, New York City) casts a shadow 1240 ft. long at the same time that a boy 5 ft. tall casts a shadow 8 ft. long. What is the height of the building?

17. The table below gives the annual cost of premium per $1000 life insurance, at various ages.

Age	21	25	30	35	40	45	50
Premium	18.25	20.04	22.60	26.40	30.50	36.10	45.20

Show this graphically. Measure *age* along the horizontal axis. What would probably be the cost at age 28? at 42?

18. In how many years will $450 earn $67.50 interest at 6 %?

19. If you know an acute angle, and one side, of a right triangle, can you find either of the other sides? Illustrate.

20. Which is the greater, $5x^2$ or $(5x)^2$?

CHAPTER XI

HOW TO SOLVE GRAPHICALLY EQUATIONS WHICH CONTAIN TWO UNKNOWNS

Section 105. Importance of skill in drawing the picture or graph of an equation. In Chapter VIII we learned how to represent and to determine the **relationship** between quantities that change together. Three methods of doing this were studied: (1) the tabular method; (2) the graphic method; (3) the equational or formula method. One of the most important facts for us to recall is that the graph and the equation **tell exactly the same thing**. For example, on page 151, the line BC and the equation $C = .12n$ tell exactly the same thing. Any information that you get from the equation you can also get from the graph. *Furthermore, relationships can be seen more easily from graphs than from tables or equations.* For these reasons, and since much of our later work in mathematics makes use of graphic methods, we need to be skillful in drawing the line which stands for an equation.

Section 106. We need to know how to locate or to "plot" points. But a line may be regarded as a series of points. Thus, to represent or locate a line we have to locate a series of its points. It happens that much of our elementary work, furthermore, deals with *straight* lines. This kind of line, clearly, can be fully determined by locating any **two** of its points.

HOW TO LOCATE OR PLOT POINTS

Thus, we see that the important thing in "graphing" is how to locate, or to represent, points. In every graph that you have already constructed you have had to *locate points* through which to draw the line. For example, in constructing a cost graph it is necessary to locate several points

representing the cost of different numbers of units of the article. Let us study more carefully how points are located.

Section 107. How points are located on maps. (1) Points are located on maps by means of latitude and longitude. Any point on the earth's surface is *definitely* located by stating its distance *east* or *west* of the prime meridian, and its distance *north* or *south* of the equator.

Thus, to the nearest degree, the location of New York is 74° W. and 41° N. because it is 74° west of the prime meridian and 41° north of the equator. Similarly, the position of Chicago is 88° W. and 42° N.; that of Paris, 2° E. and 49° N.

(2) This same method is used by many cities in numbering their houses. Two streets, which make right angles with each other, are selected as reference streets. Any house or building is completely located, then, by stating the number of blocks it is east or west, and north or south, of these reference streets.

Section 108. How points are located on drawings. By a method similar to that above, we locate points on paper. Instead of using the equator and the prime meridian as our reference lines, we **take two lines**, — for convenience, one *horizontal* and the other *vertical*, — which make a right angle with each other. Any point may be located, then, by stating its distance to the *right* of, or to the *left* of, the vertical reference line; and its distance *above* or *below* the horizontal reference line.

Thus, in Fig. 127, point A is 2 units to the *right* of, and 1 unit *above*, the reference lines; point B is 2 units to the *left* of, and 3 units *below*, the reference lines; point C is 3 units to the *left* of, and 2 units *below*, the reference lines; and point D is 0 units to the right or left of, and 3 units *above*, the reference lines.

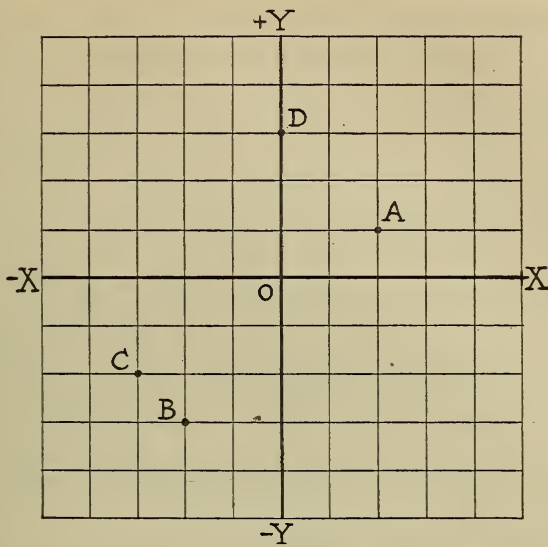


FIG. 127

Section 109. The point from which distances are measured: The origin. The point in which the two axes meet, or their intersection point, is called the **origin.** It is the point from which we measure distances, either way. The origin is usually lettered with a capital *O*, as in Fig. 127.

Section 110. How distances are distinguished from each other. It would be laborious to state that a particular point is "to the right of" or "to the left of" some reference line, each time we refer to it. To avoid this, it has been agreed to call *distances to the right of the Y-axis* ***positive***, and *distances to the left of the Y-axis* ***negative.*** Similarly, *distances above the X-axis are* ***positive***, and *distances below it are* ***negative***. It is very important to remember these facts, as we use them so often in graphic work.

Thus, in Fig. 127, the position of point *A* is described by

the numbers $+2$ and $+1$, or by $(2, 1)$. This means that point A is 2 units to the right of the Y-axis, and 1 unit above the X-axis. Similarly, the position or location of point B is described by the numbers -2 and -3, or by $(-2, -3)$; this means that point B is 2 units to the left of the Y-axis and 2 units above the X-axis. In the same way, the position of point C is described by the numbers -3 and -2, or $(-3, -2)$; this means that point C is 3 units to the left of the Y-axis and 2 units below the X-axis.

At this time the student should note that in stating the location of a point, its distance to the right of, or to the left of, the Y-axis is *always* given *before* its distance above or *below* the X-axis. This is done to avoid confusion. *That is, the x-distance is always first*, the y-distance second. Remember that the "x-distance" means the distance to the right or the left of the vertical axis.

Section 111. Plotting a point. By "plotting a point" we mean the locating, on cross-section paper, of a point whose x-distance and y-distance are known.

Thus, to plot A, whose x-distance is $+3$ and whose y-distance is $+4$, usually written $(3, 4)$, means to locate on the cross-section paper a point 3 units to the *right* of, and 4 units *above*, the origin, as in Fig. 128. In the same way, the point $(-2, 1)$ is point B on the graph.

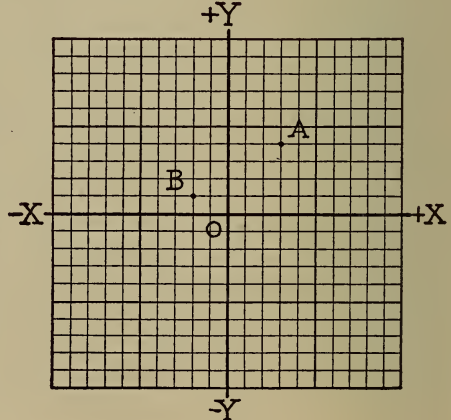


FIG. 128

EXERCISE 92

PRACTICE IN PLOTTING POINTS

1. The x-distance of a point is $+3$, *i.e.* it is 3 units to the right of the vertical axis. Is it **definitely** located? Why?
2. The y-distance of a point is -4, *i.e.* it is 4 units below the horizontal axis. Is it **definitely** located? Why?
3. A certain point is on both axes. What are its x- and y-distances?
4. Plot the points whose position is determined by the following: $(4, 2)$, $(5, 6)$, $(-3, 2)$, $(-4, -1)$, and $(-6, 2)$.
5. Plot the following: $(2, 8)$, $(3, 7)$, $(4, 6)$, $(5, 5)$, $(6, 4)$, $(8, 2)$, $(10, 0)$.
6. Plot the following: $(12, -2)$, $(15, -5)$, $(18, -8)$, $(10, 0)$, $(5, 5)$.
7. Plot the following: $(2\frac{1}{2}, 3)$, $(1\frac{3}{4}, 5)$, $(-2\frac{1}{3}, 3)$.

HOW TO DRAW THE GRAPH OF AN EQUATION WHICH CONTAINS TWO UNKNOWNS

Section 112. The picture of an equation. Now that we have learned how to locate, or plot, points, we come to the main purpose of the chapter: **to show how equations can be solved graphically.**

First illustrative example. Let us take an equation which contains two unknowns, such as,

$$y = 2x + 3.$$

In this equation the value of y changes as the value of x changes. Clearly, the value of y *depends upon* the value of x. For example, if $x = 1$, then $y = 5$; if $x = 2$, then $y = 7$, etc. A table will help to show this *relation* between the unknowns, x and y.

The equation is $y = 2x + 3$.

TABLE 18

If x equals	1	2	3	4	5	0	-1	-2	-3	-4	-5
then y equals	5	7	9	11	13	3	1	-1	-3	-5	-7

If we select any particular value of x, and the corresponding value of y which accompanies it, such as 1 and 5, or 2 and 7, we may think of them as completely describing the position of points on a graph. Thus, (1, 5), (2, 7), (3, 9), etc., *definitely locate the position of the points.* Plotting these points with respect to an X- and Y-axis, we get a series of points, such as Fig. 129. By joining these points we obtain a straight line, *which is the picture or the graphical representation of the equation* $y = 2x + 3$.

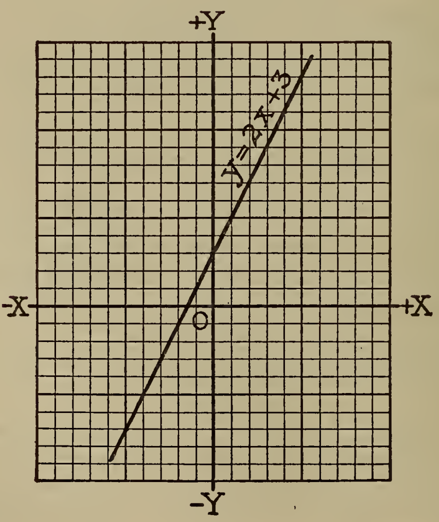


FIG. 129

Second illustrative example. For a second example, let us consider an equation in two unknowns which shows that the sum of two numbers is always 10, such as $x + y = 10$. It is clear that one number, x, *might* be 2, and if so, that y *must* be 8; or that x *might* be 4, and if so, the other number, y, *must* be 6. Thus the two unknowns, *x and y, may have many different values.* A table helps to show this.

The equation is $x + y = 10$.

TABLE 19

If x equals	1	2	3	4	0	-1	-2	-3
then y equals	9	8	7	6	10	11	12	13

Now we may think of any pair of related numbers, such as 1 and 9, or 2 and 8, as describing the position of points on this line. Thus, (1, 9), (2, 8), (3, 7), (4, 6), etc., show the location of points on the graph. Plotting these points, we have Fig. 130. By joining these points we obtain in this illustrative example the "graph" or picture of the equation $x + y = 10$. Expressed in another way, we have represented graphically the relation between two numbers whose sum is always 10.

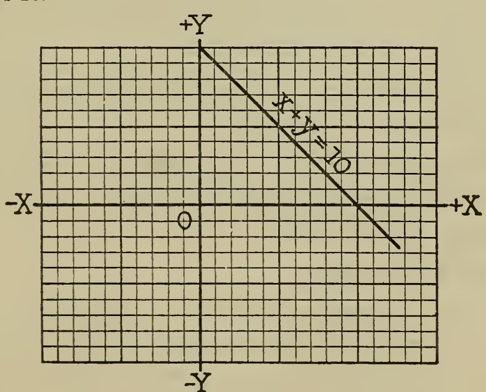


FIG. 130

EXERCISE 93

PRACTICE IN REPRESENTING GRAPHICALLY EQUATIONS WHICH CONTAIN TWO UNKNOWNS

1. In the equation $y = 2x + 5$, find the value of y when $x = 1$; when $x = 2$; when $x = 4$; when $x = 0$; when $x = -3$. Make a table similar to the one above, showing these related pairs of numbers.
2. In the equation $x = y - 4$, find the value of x when $y = 0$; when $y = 1$; when $y = 4$; when $y = 6$; when $y = -2$; when $y = -6$. Tabulate.
3. Plot the equation given in Example 2.
4. Plot the equation $x + y = 6$. HINT: First tabulate related values of x and y. Use only four points.
5. Plot, or graph, $y = 5 + x$.
6. Plot $2x + y = 6$, or $y = 6 - 2x$.
7. Graph $x - 2y = 5$, or $x = 2y + 5$.
8. Plot $3x - y = 8$, or $y = 3x - 8$.
9. Show the graphical representation of two numbers whose sum is always 10; *i. e.* of $3 + y = 10$.
10. Show the graphical representation of two numbers whose difference is always 8.
11. Graph $3x + y = 12$.
12. Plot $8 = x - 2y$.
13. If you wanted to graph the equation $2x + 4y = 12$, would you tabulate it in this form, or would you rewrite it for greater convenience in tabulating?

14. Graph $\frac{1}{2}x = \frac{1}{3}y - 8$.

15. Graph $3x = y - 2$.

Section 113. An easier method of plotting a line. A straight line is definitely determined or located if any *two* of its points are known. If these points are not too close together, they fix the plotted position of the line just as accurately as eight or ten points. Therefore, in plotting a straight line, it is sufficient to **plot only two points**, unless they are quite close together.

The easiest points to plot are those on the axes; that is, the points where the line cuts the x-axis and the y-axis. By referring to Fig. 130, or to the graph of any line, you will see that **the x-distance of the point in which the line cuts the vertical, or y-axis, is always 0, and that the y-distance of the point in which the line cuts the horizontal, or x-axis, is always 0.** Thus, if we let x be 0 in any equation, such as $2x - y = 6$, we find the point in which the line cuts the y-axis. If x is 0 in $2x - y = 6$, we see that y equals -6, **which shows that the line cuts the y-axis at a point $(0, -6)$; that is, 6 units below the origin.** In the same way, if we let y be 0 in any equation, we find the point in which the line cuts the x-axis. In this particular equation, $2x - y = 6$, if y is 0, then x is 3, which shows the point in which the line $2x - y = 6$ cuts the x-axis.

This shorter method requires only the following brief table:

TABLE 20

x equals	0	?
y equals	?	0

EXERCISE 94

1. If $x=0$, what is y in the equation $2x+y=8$? What is x if $y=0$? From these two sets of values for x and y, plot the equation.
2. Given $4x-2y=8$. Plot by finding where the line cuts the axes.
3. Where does the graph of $5x+2y=10$ cut the x-axis? the y-axis?
4. Where does the graph of $2x-3y=-6$ cut the x-axis? the y-axis? Plot.
5. Graph $2\frac{1}{2}x+y=5$.
6. Plot $3x+2y=12$.

HOW TO SOLVE GRAPHICALLY EQUATIONS WITH TWO UNKNOWNS

Section 114. When is an equation with two unknowns solved? In equations with only one unknown, such as $3x+4=19$, we found that there was only *one* value for x which would satisfy the equation; namely, $x=5$. If we substitute 5 for x in this equation, giving $15+4=19$, we find that 5 **satisfies the equation.** Any other number substituted for x would not "satisfy the equation."

But now consider an equation which has *two* unknowns, such as

$$x+y=8.$$

Here we see that x *might* be 3 and y *would* be 5; or x *might* be 6 and y *would* be 2; or x *might* be 10 and y *would* be -2. Thus, there are a great **many sets of values of x and y which could satisfy the equation** $x+y=8$. This will be made clear as you work the following examples.

EXERCISE 95

1. Give four sets of values of x and y that will satisfy the equation $x-y=6$.
2. Will $x=4\frac{1}{2}$ and $y=3$ satisfy the equation $4x-y=15$? Does $x=5$ and $y=4$ satisfy it?
3. If the equation $x+y=8$ is plotted, would the points (5, 3) lie on the line representing the equation? (3, 4)? (10, −2)? (1, 7)?
4. Does the graph of the equation $2x-y=7$ pass through the point (5, 3)? (4, 2)?

We have shown that an equation with two unknowns can be satisfied by a very large number of pairs of values of the unknowns. Each pair of values that satisfies the equation is called a **solution** of the equation. Therefore, "solving the equation" means finding a pair of values that will satisfy the equation.

Section 115. Linear equations. The fact that the graph of an equation which contains two unknowns, *each of the first degree* (*i.e.* no squares or cubes), is **always** a straight line, has led to the name **linear equations.** Thus, $2x+y=5$, $x+5=10$, etc., are **linear equations.**

TWO LINEAR EQUATIONS MAY BE EASILY SOLVED BY PLOTTING THEM ON THE SAME AXES

Section 116. It is a very common problem in mathematics to have to find **one set of values** which will satisfy each of two equations having two unknowns. For example, what single set of values will satisfy each of these equations?

$$\begin{cases} x+y=8 \\ 2x-y=7 \end{cases}$$

It is clear that $x = 6$ and $y = 2$ or (6, 2) will satisfy the first equation, but not the second one; in the same way $x = 4$ and $y = 4$ or (4, 4) satisfies the first equation, but not the second one; $x = 6$ and $y = 5$ satisfies the second equation, but not the first one.

OUR PROBLEM IS TO FIND ONE SET OF VALUES THAT WILL SATISFY BOTH EQUATIONS

This can be done, *graphically*, by plotting both equations on the same axes, because in that way we can find a point common to the two lines; that is, the **point in which two lines intersect. The coördinates of this point will satisfy both equations.** Figure 131, on the following page, shows both equations plotted on the same axes. Note that the two lines intersect at the point (5, 3). **This point of intersection of the two lines gives a single set of values,** $x = 5$ **and** $y = 3$, **which satisfies both equations.** (Show that $x = 5$ and $y = 3$ checks for each of the equations.)

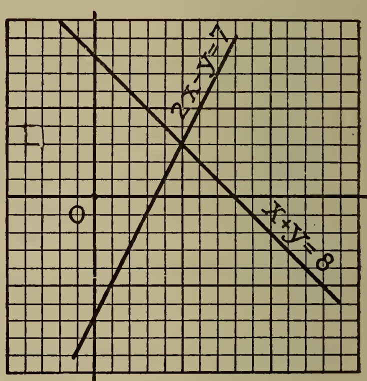


FIG. 131

EXERCISE 96

Find a set of values that will satisfy each of the following pairs of equations, by finding the intersection point of their graphs. Check by substituting in the equations.

1. $\begin{cases} x + y = 6 \\ 2x - y = 3 \end{cases}$

2. $\begin{cases} y = 2x + 3 \\ y = x + 7 \end{cases}$

3. $\begin{cases} x-y=5 \\ 2x+y=7 \end{cases}$

4. $\begin{cases} 2x+y=12 \\ x-2y=-4 \end{cases}$

5. $\begin{cases} y=x+6 \\ y=-x+2 \end{cases}$

6. $\begin{cases} a+b=7 \\ a-2b=-5 \end{cases}$

7. $\begin{cases} x+y=1 \\ x-y=5 \end{cases}$

8. $\begin{cases} y=\frac{1}{2}x+5 \\ x=\frac{1}{3}y \end{cases}$

9. $\begin{cases} x-y=8 \\ 2x+y=7 \end{cases}$

10. $\begin{cases} y=x+10 \\ x-y=-10 \end{cases}$

11. $\begin{cases} 2x+y=-11 \\ y-x=1 \end{cases}$

12. $\begin{cases} x-2y=2 \\ y=\frac{2}{3}x \end{cases}$

13. $\begin{cases} y=x-3 \\ 2x-y=11 \end{cases}$

14. $\begin{cases} a+b=2 \\ a-2b=14 \end{cases}$

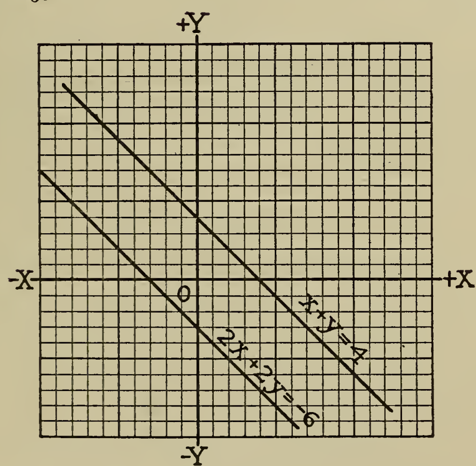


FIG. 132

Section 117. Equations whose graphs are parallel lines, *i.e.* inconsistent equations. Figure 132 shows the graphs of the equations represented below.

$$\begin{matrix} (1) \\ (2) \end{matrix} \begin{cases} x+y=4 \\ 2x+2y=-6 \end{cases}$$

Note that the lines do not intersect, but are parallel. What single set of values of x and y will satisfy each of these equations? Evidently there is none, **for they have no point in common.** Such equations are generally called **inconsistent**, to distinguish them from the kind that are satisfied by some set of values. The latter kind, those whose graphs intersect, are often called **SIMULTANEOUS EQUATIONS.**

EXERCISE 97

Graph each of the following pairs of equations to determine which pairs are inconsistent and which are simultaneous:

1. $\begin{cases} y - x = 4 \\ x - 6 = y \end{cases}$

2. $\begin{cases} 2x - 3 = y \\ 2y + 10 = 4x \end{cases}$

3. $\begin{cases} x + y = 4 \\ x - y = 6 \end{cases}$

4. $\begin{cases} y + 4 = x \\ x + y = 12 \end{cases}$

5. $\begin{cases} x - y = 10 \\ 2x + y = 2 \end{cases}$

6. $\begin{cases} y + 2 = x \\ x + y = 8 \end{cases}$

7. $\begin{cases} x = y - 3 \\ 2y = 2x + 12 \end{cases}$

8. $\begin{cases} 2x - 4 = y \\ 3y + 12 = 6x \end{cases}$

9. $\begin{cases} x - 8 = y \\ x - y = 4 \end{cases}$

10. $\begin{cases} x = 6 - y \\ y = 10 - x \end{cases}$

SUMMARY OF CHAPTER XI

In this chapter we have learned:

1. **How to locate points on maps and drawings by plotting them at given distances from a vertical and a horizontal reference line. These reference, or base lines, intersect in a point called the origin, from which we always plot and record our dis-**

tances. The X-distance on the horizontal axis is positive when plotted to the right of the vertical axis, and negative when plotted to the left; similarly the Y-distance is positive above the horizontal axis and negative below it.

2. The main purpose was to show how to draw the picture or graph of an equation by plotting a series of points which will satisfy the x and y values of the equation.

3. To plot a straight line only 2 points need be located.

4. The easiest points to locate are those in the axes. For these either x or y is 0.

5. Two equations may be solved graphically by plotting the two lines. The values of x and y can be found by reading the x and y distances of this point.

REVIEW EXERCISE 98

1. From the sum of -6 and $+10$ take -8.

2. The product of two numbers is $-40y$; one of them is $+10$. What is the other?

3. State the four principles, or axioms, used in solving equations. Illustrate in solving the equation $5y - 8 = +2y - 50$.

4. If $A = 4x + 3y$ and $B = 4x - 3y$, what does $A + B$ equal? What does $A - B$ equal?

5. Does $\frac{5}{7} = \frac{5 \cdot 2}{7 \cdot 2}$? Does $\frac{a}{b} = \frac{ac}{bc}$? Does $\frac{1}{2} = \frac{5}{10}$?

State the principle involved in these examples.

Age	Boys who leave school at the age of 14 earn weekly wages as indicated	Boys who leave school at the age of 18 earn weekly wages as indicated
14	$ 4.00	0
16	5.00	0
18	7.00	$10.00
20	9.50	15.00
22	11.00	20.00
24	12.00	24.00
25	13.00	30.00

6. Studies have been made to determine the money value of a high school education. The table above shows the average weekly earnings for boys who leave school at the age of 14, and for those who remain in school until they are 18 years old.

   Graph the earnings for each class of boys on the same axes. Measure *age* along the horizontal axis.

   Interpret the graph. If a boy knew that he would live to be only 25 years old, would it pay him, in dollars, to go to high school? How much?

7. In solving a particular problem, how do you tell whether to use the cosine, or tangent? Illustrate by specific examples.

CHAPTER XII

HOW TO SOLVE EQUATIONS WITH TWO UNKNOWNS BY ALGEBRAIC METHODS

SOLUTION BY ELIMINATING ONE UNKNOWN

Section 118. The need for a shorter method of solving equations with two unknowns. In the previous chapter we saw that equations with two unknowns can be solved by graphic methods. The exclusive use of that method, however, would require a great deal of time, and would necessitate that we have cross-section paper at all times. Fortunately, there is **a shorter method** which can be used. This is a method by which we **ELIMINATE ONE OF THE UNKNOWNS.** By eliminating or getting rid of **one** of the unknowns, we obtain an equation with only **one** unknown. The following illustrative examples will explain the different ways by which one of the unknowns is eliminated. This chapter will show two methods of elimination.

I. ELIMINATION OF ONE UNKNOWN BY ADDING OR BY SUBTRACTING THE MEMBERS OF THE EQUATIONS

Section 119. A great many equations in two unknowns can be most easily solved by this method. The following example illustrates it.

Illustrative example. Find the value of x and y in these equations:

$$\begin{cases} 2x - y = 5, & (1) \\ x + y = 13. & (2) \end{cases}$$

Adding equation (1) and equation (2) gives

$$3x = 18, \qquad (3)$$

or

$$x = 6. \qquad (4)$$

Substituting 6 for x in (1) and (2) gives

$$y = 7. \qquad (5)$$

Check: Substituting 6 for x and 7 for y in (1) and (2) gives

$$12 - 7 = 5. \tag{6}$$
$$6 + 7 = 13. \tag{7}$$

It happens in this example that one of the unknowns, y, is eliminated by adding the corresponding members of the given equations. In many examples it is possible to eliminate one of the unknowns by subtracting the members of one equation from the corresponding members of the other. In many other examples, however, it is impossible to eliminate one of the unknowns directly, either by adding or by subtracting the members of the two equations. For illustration, take this set of equations:

Illustrative example.

$$\begin{cases} x - 2y = 8, & (1) \\ 2x + y = 6. & (2) \end{cases}$$

If we add the corresponding members of the two equations, we get the equation $3x - y = 14$. But this does *not* eliminate either of the unknowns. In the same way, if we subtract (2) from (1), we get the equation $-x - 3y = 2$. Again, this does *not* eliminate either one of the unknowns. This shows that addition or subtraction of the members to the equations will *not* eliminate one of the unknowns *unless* one of them, the x or the y, has the same *coefficient* in both equations.

Now let us *make* y in the second equation have the same coefficient as y in the first equation. To do so, the second equation must be multiplied through by 2. This gives

$$\begin{cases} 4x + 2y = 12, & (3) \\ x - 2y = 8. & (1) \end{cases}$$

Now, by adding (3) and (1), we get rid of y, obtaining:

$$5x = 20, \tag{4}$$

or

$$x = 4. \tag{5}$$

Substituting in (1) or (2), $y = -2$. (6)

Check: $16 - 4 = 12$. (7)

$4 + 4 = 8$. (8)

An important question naturally arises here: When do we eliminate by *addition* and when by *subtraction?* This can be answered by referring to an example.

$$\begin{cases} x + y = 11, & (1) \\ 2x + y = 4. & (2) \end{cases}$$

Would either x or y be eliminated by adding the corresponding members of these equations? Certainly not, for that would give $3x + 2y = 15$. Now, would either x or y be eliminated by subtracting the members of one equation from those of the other? Yes, for we should have $-x = 7$. However, if the second equation (2) above had been $2x - y = 4$, then we should eliminate y by *adding* (1) and (2).

From these examples we come to the following conclusions about eliminating one of the variables:

I. If the variable we wish to eliminate has the same sign in both equations, then it is eliminated by subtracting the members of one equation from the members of the other equation.

II. If the variable we wish to eliminate has different signs in the two equations, it is eliminated by adding the corresponding members of the equations.

III. No variable can be eliminated either by addition or by subtraction unless it has the same *coefficient* in both equations. If it does not have the same coefficient in both equations, then we must multiply the sides of one, or both, of the equations by such a number, or numbers, as

will make that variable have the same coefficient. Thus, to eliminate x in the following equations:

$$\begin{cases} 2x - y = 8, & (1) \\ 3x + 4y = 23. & (2) \end{cases}$$

It is necessary to multiply (1) by 3 and to multiply (2) by 2. This gives the following equations:

$$\begin{cases} 6x - 3y = 24, & (3) \\ 6x + 8y = 46. & (4) \end{cases}$$

Now the variable x can be eliminated by subtracting (4) from (3), which gives

$$-11y = -22,$$

or

$$y = 2.$$

EXERCISE 99

ELIMINATION BY ADDITION OR SUBTRACTION

Solve and check each of the following:

1. $\begin{cases} x + y = 5 \\ x - y = 2 \end{cases}$

2. $\begin{cases} 2x + 3y = 14 \\ 3x - 3y = 1 \end{cases}$

3. $\begin{cases} 2r + s = 9 \\ r - s = 12 \end{cases}$

4. $\begin{cases} x - y = -8 \\ x + y = -4 \end{cases}$

5. $\begin{cases} 3a - b = -2 \\ 4a + b = -12 \end{cases}$

6. $\begin{cases} 2x - y = 11 \\ x - 3y = 13 \end{cases}$

7. $\begin{cases} 3b + 2c = 5 \\ 2b + c = 3 \end{cases}$

8. $\begin{cases} 4x - 3y = 8 \\ x - 7y = 2 \end{cases}$

9. Find two numbers whose sum is 100 and whose difference is 18.

10. In an election of 642 votes an amendment was carried by a majority of 60 votes. How many voted *yes* and how many *no?*

11. The admission to a school play was 25 cents for adults and 15 cents for children. The proceeds

from 267 tickets were \$50.05. How many tickets of each kind were sold?

12. A purse containing 18 coins, dimes and half dollars, amounts to \$6.20. Find the number of each denomination.

13. $\begin{cases} p - q = 8 \\ 3p + 4q = 10 \end{cases}$

14. $\begin{cases} \frac{1}{2}x + y = 3 \\ 2x + 3y = 11 \end{cases}$

15. $\begin{cases} 2x + 5y = 12 \\ y - 3x = -1 \end{cases}$

16. $\begin{cases} x + 2y = 11 \\ 5x - 3 = 3y \end{cases}$

17. $\begin{cases} 3x - y = -2 \\ x - 3y = 10 \end{cases}$

18. $\begin{cases} 2a - 8 = -b \\ 3a + 4b = 7 \end{cases}$

19. $\begin{cases} \frac{1}{2}x + 3 = -4y \\ x - \frac{1}{2}y = 11 \end{cases}$

20. $\begin{cases} \frac{3}{4}x + \frac{2}{3}y = 12 \\ \frac{1}{2}x - \frac{4}{3}y = -8 \end{cases}$

In the remaining examples of this exercise, use any method of elimination.

21. $\begin{cases} y = x + 4 \\ 2x + y = 16 \end{cases}$

22. $\begin{cases} 2x = 3y + 24 \\ x - y = 10 \end{cases}$

23. $\begin{cases} \frac{1}{2}x - \frac{1}{5}y = 9 \\ x = y + 2 \end{cases}$

24. $\begin{cases} 4x = 3y + 3 \\ 5y = 6x \end{cases}$

25. $\begin{cases} x = -2y - 1 \\ y = x + 14.5 \end{cases}$

26. $\begin{cases} b = \frac{1}{2}c \\ 2c - 3b = 7 \end{cases}$

27. $\begin{cases} \frac{2}{3}x - \frac{3}{4}y + 1 = 0 \\ \frac{1}{4}y + x = -15 \end{cases}$

28. $\begin{cases} 2x + 2y = 0 \\ x = y + 12 \end{cases}$

II. ELIMINATION BY SUBSTITUTION; THAT IS, BY SUBSTITUTING THE VALUE OF X FROM ONE EQUATION IN THE OTHER EQUATION

Section 120. This method of elimination will be illustrated by working two easy examples.

Illustrative example. Find the value of x and of y in the following equations:

$$\begin{cases} x + y = 10, & (1) \\ x - 3y = -6. & (2) \end{cases}$$

Solving equation (1) for x, we get

$$x = 10 - y. \qquad (3)$$

Substituting $10 - y$ for x in (2) gives

$$10 - y - 3y = -6, \qquad (4)$$

or $$-4y = -16, \qquad (5)$$

or $$y = 4. \qquad (6)$$

Substituting 4 for y in (1) or (2) gives

$$x = 6. \qquad (7)$$

Here, as when we eliminate one unknown by adding or subtracting equations, our real aim is to get an equation which contains only *one* unknown. We found from equation (1) that $x = 10 - y$. This value of x must be true for both equations. (Recall that x is the same for both equations, or for both lines, **at their point of intersection.**) For this reason we may substitute $10 - y$ in place of x in the second equation. This gives an equation in one unknown; namely, y.

The same results could have been obtained by finding the value of y, instead of the value of x, from one of the equations and substituting it in the other equation. For example:

Second illustration of the method of eliminating by substitution.

$$\begin{cases} x + y = 10, & (1) \\ x - 3y = -6. & (2) \end{cases}$$

From equation (1), $$y = 10 - x. \qquad (3)$$

Substituting $10 - x$ for y in (2)

$$x - 3(10 - x) = -6, \qquad (4)$$

or $$x - 30 + 3x = -6, \qquad (5)$$

or $$4x = 24, \qquad (6)$$

or $$x = 6, \qquad (7)$$

and $$y = 4, \text{ as before.}$$

EXERCISE 100

Solve by the method of substitution and check each result:

1. $\begin{cases} x-2y=10 \\ 3x+2y=6 \end{cases}$

2. $\begin{cases} x-3y=-1 \\ 4x-y=-15 \end{cases}$

3. $\begin{cases} 5r-4s=18 \\ r=2s+7 \end{cases}$

4. $\begin{cases} y=x \\ 3x+4y=7 \end{cases}$

5. $\begin{cases} x+y=8 \\ 2x-y=10 \end{cases}$

6. $\begin{cases} a-2b=-13 \\ b-a=9 \end{cases}$

7. The sum of two numbers is 102; the greater exceeds the smaller by 6. Find the numbers.

8. The difference between two numbers is 14, and their sum is 66. Find the numbers.

9. 12 coins, nickels and dimes, amount to $1.05. Find the number of each kind of coin.

10. The perimeter of a rectangle is 158 inches; the length is 4 feet more than twice the width. Find the dimensions of the rectangle.

11. Bacon costs 10 cents per pound more than steak. Find the cost per pound of each if 4 pounds of bacon and 7 pounds of steak together cost $3.48.

12. A part of $4000 is invested at 4% and the remainder at 5%. The annual income on both investments is $185. Find the amount of each investment.

13. The quotient of two numbers is 2, and the larger exceeds the smaller by 7. Find the numbers.

14. Oranges cost 20 cents per dozen more than apples. A customer bought 10 dozen oranges

and 4 dozen apples and received 20 cents in change from a 5-dollar bill. Find the price per dozen of each.

15. $\begin{cases} 2x + 3y = 5 \\ 4x - y = 3 \end{cases}$

16. $\begin{cases} \frac{1}{2}p - \frac{2}{3}q = -1 \\ \frac{3}{4}p = q - \frac{3}{2} \end{cases}$

17. $\begin{cases} \dfrac{x}{2} - \dfrac{y}{3} = 9 \\ \dfrac{2x}{3} - \dfrac{3y}{4} = \dfrac{7}{3} \end{cases}$

13. $\begin{cases} \frac{1}{6}x - \frac{1}{9}y = 1 \\ \frac{1}{2}y - 2x = 3 \end{cases}$

EXERCISE 101

Solve by either method and **check** each of the following:

1. $\begin{cases} x = 2y - 3 \\ x = 5y - 21 \end{cases}$

2. $\begin{cases} x - y = 10 \\ x = 16 - 2y \end{cases}$

3. $\begin{cases} y + 2x = 12 \\ 5x + y = 42 \end{cases}$

4. $\begin{cases} s - 3t = -3 \\ 4s + t = 14 \end{cases}$

5. $\begin{cases} 2b + 3c = 6 \\ b = 5c + 16 \end{cases}$

6. $\begin{cases} x + 5y = 1 \\ 2x + 6y = -2 \end{cases}$

7. The sum of two numbers is 14; the larger exceeds the smaller by 2. Find each number.

8. Twenty coins, dimes and nickels, have a value of \$1.70. Find the number of each.

9. A boy earns \$2 per day more than his sister; the boy worked 8 days and the girl worked 6 days. Both together earned \$44. What did each earn per day?

10. $\begin{cases} y = \frac{3}{2}x - 5 \\ y + x = 10 \end{cases}$

11. $\begin{cases} 2x + 3y = 5 \\ 3x - y = 2 \end{cases}$

12. $\begin{cases} \frac{1}{2}x = 10 - \frac{1}{3}y \\ \frac{2}{3}x = 14 - \frac{1}{2}y \end{cases}$

13. $\begin{cases} y = 2x - 10 \\ x = 2y - 14 \end{cases}$

14. Three tons of hard coal and two tons of soft coal cost \$42. At the same prices, 2 tons of hard coal and 3 tons of soft coal would cost \$38. What is the price per ton of each kind of coal?

15. The base of one rectangle is 8, and the base of a smaller rectangle is 6. The sum of their areas is 134, and the difference between their areas is 26. Find the height of each rectangle.

16. A man bought a farm. If he had paid \$15 less per acre, he could have bought 10 acres more; at \$25 more per acre, he could have bought 10 acres less. How many acres did he buy, and what did he pay per acre?

17. The difference between two numbers is 6; one half of the smaller number equals one third of the larger. What are the numbers?

18. If oats are worth 50¢ a bushel, and corn 75¢, how many bushels of each would you have to use to make a mixture of 80 bushels worth 60¢ a bushel?

19. A man has \$10,000 invested. Part of the money earns 6% interest, and the remainder earns 5%. Find the amount invested at each rate, if the total yearly income is \$570.

20. $\begin{cases} \frac{2}{3}x - \frac{1}{2}y = 13 \\ \frac{1}{4}x + \frac{1}{5}y = 1 \end{cases}$

21. $\begin{cases} \frac{1}{2}(x+10) - y = -4 \\ \dfrac{3}{4}x - \dfrac{2(y+2)}{4} = -10 \end{cases}$

22. $\begin{cases} \dfrac{x+6}{3} + \dfrac{y-2}{2} = 4 \\ \dfrac{4}{3}x - 5 = \dfrac{y-6}{2} \end{cases}$

23. $\begin{cases} 2(x+y) - (x-y) = 26 \\ 3(x-y) - 2(x+y) = -22 \end{cases}$

STANDARDIZED PRACTICE EXERCISE D

Practice on these examples until you can reach the standard, 6 right in 8 minutes. Record your score on your record card.

1. $\begin{cases} 3x + y = 14 \\ 6x - 3y = 3 \end{cases}$	**7.** $\begin{cases} 2a + b = 12 \\ 5a - 3b = 19 \end{cases}$ . . . .
2. $\begin{cases} 2p = 3t + 7 \\ 5p + 4t = 29 \end{cases}$	**8.** $\begin{cases} 3x = 2y + 10 \\ 7x + 3y = 31 \end{cases}$ . . . .
3. $\begin{cases} \frac{x}{2} + \frac{2y}{3} = -9 \\ 2x = 3y + 15 \end{cases}$	**9.** $\begin{cases} \frac{p}{2} + \frac{3t}{5} = -8 \\ 6p = 5t + 26 \end{cases}$ . . . . .
4. $\begin{cases} 4b + c = 18 \\ 3b - 2c = 8 \end{cases}$	**10.** $\begin{cases} 2x + y = 14 \\ 5x - 2y = 26 \end{cases}$ . . . .
5. $\begin{cases} 5x = 3y + 27 \\ 3x + 2y = 20 \end{cases}$	**11.** $\begin{cases} 4a = 3b + 9 \\ 3a + 2b = 11 \end{cases}$ . . . .
6. $\begin{cases} \frac{t}{3} + \frac{2s}{5} = -3 \\ 4t = 5s + 13 \end{cases}$	**12.** $\begin{cases} \frac{x}{2} + \frac{2y}{3} = -3 \\ 2x = 3y + 5 \end{cases}$ . . . . .

SUMMARY OF CHAPTER XII

1. **We need shorter methods (than the graphing ones) of solving equations with two unknowns.**
2. **A valuable short method is to eliminate one of the unknowns.**
3. **We have learned two ways of doing this: first, by adding or subtracting the equations; second, by substituting the value of *x* from one equation in the other equation.**

REVIEW EXERCISE 102

1. If one tablet costs b dollars, what will x tablets cost?
2. If a books cost b dollars, what will one book cost? c books?
3. What is the perimeter of a rectangle whose width is a and whose length is b? the area?
4. What is the width of a rectangle whose perimeter is p and whose length is x?
5. The area of a triangle is k. Its base is b. What is its altitude?
6. The sum of two numbers is s. If one is d, what is the other?

Solve each of the following pairs of equations by any method of elimination:

7. $\begin{cases} x - 2y = 8 \\ 3x + 2y = 8 \end{cases}$

8. $\begin{cases} a - 2b = -1 \\ 4a - b = 10 \end{cases}$

9. $\begin{cases} \dfrac{5x}{6} + \dfrac{y}{4} = 7 \\ \dfrac{2x}{3} - \dfrac{y}{8} = 3 \end{cases}$

10. The table below shows how much money (to the nearest dollar) you would have at the end of a certain number of years if you saved 10 cents a day and deposited it in a bank which pays 3 % interest.

At the end of (years)	1	2	3	5	8	10	14	17	20
total amt. saved is	37	75	115	197	330	425	635	809	999

Represent this graphically, measuring the *time* on the horizontal axis. Estimate the total savings at the end of 4 yr.; 6 yr.; 25 yr.

11. A collection box contained 63 coins, nickels and quarters. How many of each kind were there if the total amount was \$ 8.35?

12. Is it ever possible to get from a graph information which could not be obtained from the table? Illustrate.

13. The distance from the base to the top of a hill, up a uniform incline of 40°, is 800 ft. What is the altitude of the top above the base?

CHAPTER XIII

HOW TO FIND PRODUCTS AND FACTORS

Section 121. Why you should be able to find products. Suppose you wanted to find the area of a rectangle whose

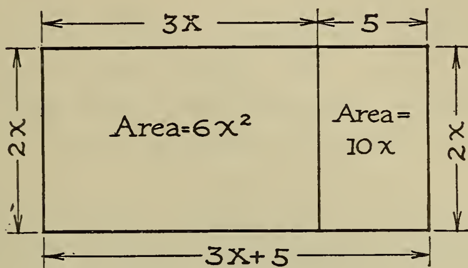


FIG. 133

dimensions are $3x+5$ and $2x$. To do so, it would be necessary to multiply $3x+5$ by $2x$, or to find the **product** of these two algebraic expressions. One way to do this is to divide the rectangle into smaller rectangles, as indicated in Fig. 133. This gives two rectangles, the dimensions of one being $2x$ by $3x$, and of the other $2x$ by 5. From what you already have learned about multiplication you can see that the areas of these are $6x^2$ and $10x$, be-

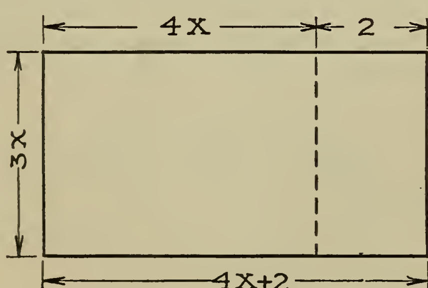


FIG. 134

cause $3x$ times $2x$ is $6x^2$ and 5 times $2x$ is $10x$. Thus the area of the original rectangle is $6x^2+10x$. Similarly, the area of the rectangle in Fig. 134 is what? What would its area be if the dimensions were $6a+4$ and $5a$?

These illustrations are given to make clear the **need of learning how to find products.** Other illustrations might have been taken. For example, what is the cost of $15b+3$ articles at $4b$ cents each? How much could you earn in $6y+4$ days at $3y$ dollars per day?

A NEW WAY OF INDICATING MULTIPLICATION

Section 122. As you progress in your study of mathematics, you will find that it has a language which tells more in fewer words or symbols than any other language. For example, instead of writing "*find the product of* $3x+5$ *and* $2x$," it has been agreed to express this by means of the parentheses, ( ). Thus, $2x(3x+5)$ means "**to find the product of $3x+5$ and $2x$,**" or, "**to multiply $3x+5$ by $2x$.**" It is **important** to note that there is no sign between the $2x$ and the expression in the parentheses. Similarly, to state algebraically the problem in the second illustration, Fig. 134, you would write $3x(4x+2)$, putting no sign between the $3x$ and the parentheses. **Thus, $5b(3b+7)$ means to multiply each of the numbers in the parentheses by $5b$.**

ORAL EXERCISE 103

PRACTICE IN FINDING PRODUCTS

In the following examples, multiply each term within the parentheses by the number which immediately precedes the parenthesis, or, remove parentheses.

Illustrative example. $3x(5x^2+7x+8)=15x^3+21x^2+24x.$

1. $4y(3y+9)$
2. $6b(2b+1)$
3. $7c(3+5c)$
4. $5x(x-4)$
5. $9(2x^2+7x-4)$
6. $4a(a^2+3a+7)$
7. $6y(3y^2-5y+2)$
8. $1(2b^2+3)$
9. $5t(6-t)$
10. $8y(y-8)$
11. $3r(r+3)$
12. $7b(1-b)$
13. $8(b^2-8b+12)$
14. $6(2a-3b+c)$
15. $ab(a+b+1)$
16. $xy^2(x+y+xy)$
17. $-1(4-5y)$
18. $-3x(6x-4)$
19. What algebraic expression will represent the area of a rectangle whose length is 10 inches more than its width?
20. What algebraic expression will represent the total daily earning of 4 men and 7 boys, if each man earns $2 per day more than each boy?
21. $-6(2x-7)$
22. $-(10-x)$
23. $-4y^2(2y-3)$
24. $(16y^2-7)$
25. $(a+b)$
26. $-(b-c)$

In this exercise you have learned how *parentheses* are used to indicate that *each* of the terms in an expression must be multiplied by another number.

Section 123. More difficult multiplication. Most products which you will need to find are more difficult than those of the preceding exercise. For example: How many square feet of floor area in a dining room whose dimensions are $4x+3$ ft. and $5x+4$ ft.? To find the **product** of these factors requires something you have not yet learned. You know how to multiply $5x+4$ by $4x$ or $4x+3$ by $5x$, but you have not learned how to multiply

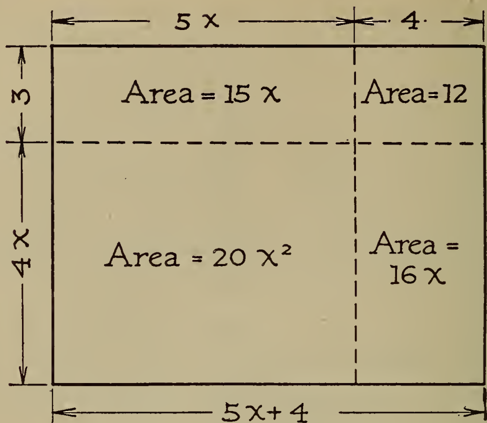


FIG. 135

such expressions as $4x + 3$ by $5x + 4$. The drawing shows one way to do this; namely, the **geometrical method** of dividing the entire area into smaller rectangles, the area of each of which you can find. This method gives four rectangles whose areas we can find. Thus, we get four rectangles whose areas are $20x^2$, $16x$, $15x$, and 12, or, **collecting terms**, an entire area of $20x^2 + 31x + 12$.

Another method of finding the product of $4x + 3$ and $5x + 4$ (which is generally written as $(4x + 3)(5x + 4)$) makes no reference to rectangles. It is very much like the method of multiplication used in arithmetic. To illustrate, this same example could be solved as follows:

First illustrative example.

$$(4x + 3)(5x + 4).$$

$$\begin{array}{r} 4x + 3 \\ 5x + 4 \\ \hline 20x^2 + 15x \\ + 16x + 12 \\ \hline 20x^2 + 31x + 12 \end{array}$$

The $20x^2 + 15x$ is the result of multiplying $4x + 3$ by $5x$, and the $16x + 12$ is the result of multiplying $4x + 3$ by 4. This latter method is much more generally used than the geometrical method. Let us take another illustration.

Second illustrative example.

$$(7a + 3)(4a - 5).$$

$$\begin{array}{l} 7a + 3 \\ \underline{4a - 5} \\ 28a^2 + 12a \\ \underline{\qquad - 35a - 15} \\ 28a^2 - 23a - 15 \end{array}$$

Note that $7a + 3$ was first multiplied by $4a$, giving $28a^2 + 12a$. Then $7a + 3$ was multiplied by -5, giving $-35a - 15$. How was the final product obtained?

Which of these two methods, do you think, should be learned?

EXERCISE 104

PRACTICE IN FINDING PRODUCTS

Illustrative example.

$$(2x^2 + 3x + 4)(3x - 7).$$

$$\begin{array}{l} 2x^2 + 3x + 4 \\ \underline{3x - 7} \\ 6x^3 + 9x^2 + 12x \\ \underline{\qquad - 14x^2 - 21x - 28} \\ 6x^3 - 5x^2 - 9x - 28 \end{array}$$

1. $(3a + 2)(4a + 5)$
2. $(7b + 5)(b + 6)$
3. $(y + 4)(5y + 3)$
4. $(t - 5)(2t + 8)$
5. $(8y + 5)(7y + 3)$
6. $(6c + 4)(4c + 6)$
7. $(4a - 3)(4a - 8)$
8. $(x + 3)(x - 9)$

9. $(3c-4)(3c-9)$
10. $(6t-7)(3t-9)$
11. $(y+6)(y+6)$
12. $(3b^2+1)(2b^2+5)$
13. $(4x^3+3)(3x^3+10)$
14. $(5a+3)(5a-3)$
15. $(7c+5)(7c-5)$
16. $(a+7)(a-7)$
17. $(3a-2)(3a-2)$
18. $(x^2+2x+1)(x-3)$
19. $(y^2+6y+9)(y+3)$
20. $(x^3+5)(x^3-5)$
21. If you should multiply $x+7$ by $x+10$, what would be the *first* term of your product? What would be the *last* term?
22. Can you tell, at a glance, the *first* and *last* terms of the product which you would obtain by multiplying $2x+7$ by $5x+4$?

A SHORTER METHOD OF MULTIPLYING

Section 124. There is a much shorter method of finding products like those in Exercise 104. Mastery of this short cut will not only save a great deal of time, but it will help you in the later work of this chapter. For an illustration, take the example

$$(3b+5)(2b+7).$$

You have no difficulty in seeing that the first term of the product is $6b^2$ (*i.e.* $3b \times 2b$) and that the last term is $+35$ (*i.e.* 5×7). So if there were some method by which you could tell the middle term, you could write the product at once, without using the longer method of placing one factor under the other and multiplying in the regular way. To make the new method clear, it is necessary to refer again to the regular way. Let us illustrate with the example:

Illustrative example.

$$(3b+5)(2b+7).$$

By the old method :

$$\begin{array}{r}
3\,b + 5 \\
\underline{2\,b + 7} \\
6\,b^2 + 10\,b \phantom{{}+35} \\
\underline{+ 21\,b + 35} \\
6\,b^2 + 31\,b + 35
\end{array}$$

The arrows show the *cross-multiplications* or cross-products that make up the middle term. One "cross-product" is $2\,b$ times $+5$, or $+10\,b$, and the other cross-product is $+7$ times $+3\,b$, or $+21\,b$. Combining the cross-products, we get the middle term, $+31\,b$.

By the shorter method we get

$$\overbrace{(3\,b + \underbrace{5)(2\,b}_{+10\,b} + 7)}^{+21\,b} = 6\,b^2 + 31\,b + 35.$$

The curved lines indicate the cross-multiplication, or **cross-products**, which must be combined to give the middle term, $+21\,b$ and $+10\,b$, giving $+31\,b$. Thus, in this new method, it is assumed that you can tell, at a glance, the first and last terms of the product. Then you can **get the middle term by finding the sum of the cross-products.** The curved lines are drawn to help you see the cross-products.

Second illustrative example.

$$(4\,b + 3)(7\,b - 8).$$

THE LONG METHOD

$$\begin{array}{r}
4\,b + 3 \phantom{{}-24} \\
\underline{7\,b - 8} \phantom{{}-24} \\
28\,b^2 + 21\,b \phantom{{}-24} \\
\underline{- 32\,b - 24} \\
28\,b^2 - 11\,b - 24
\end{array}$$

THE SHORT METHOD

$$\overbrace{(4\,b + \underbrace{3)(7\,b}_{+21\,b} - 8)}^{-32\,b} = 28\,b^2 - 11\,b - 24$$

It is important to recognize that **the curved lines indicate cross-products in just the same way that the arrows in the**

long method refer to cross-products. The new method, which from now on we shall call the **CROSS-PRODUCT METHOD**, enables you to do **mentally** in much less time what was **written down** by the old method. To give you practice in this important method of finding the product of two factors the following exercise has been included.

EXERCISE 105

Find the products of the following factors by the *cross-product* method:

1. $(x+2)(x+5)$
2. $(2y+4)(3y+5)$
3. $(b+6)(2b+7)$
4. $(x+4)(3x+5)$
5. $(t-8)(t-3)$
6. $(5b+3)(b+1)$
7. $(9x+2)(2x+1)$
8. $(4c+3)(7c+10)$
9. $(5s+3)(5s-3)$
10. $(a+9)(a+9)$
11. $(s+2)(s-5)$
12. $(ab+6)(ab+3)$
13. $(x^2+3)(x^2+6)$
14. $(abc+8)(abc-10)$
15. $(2x+3y)(2x+3y)$
16. $(a+b)(a+b)$
17. $(c+d)(c+d)$
18. $(f+s)(f+s)$
19. $(x+5)(x-5)$
20. $(x+10)(x-10)$
21. $(y+4)(y-4)$
22. $(3t+7)(3t-7)$
23. $(4a+3)(4a+3)$
24. $(7b+9y)(7b+9y)$
25. $(y+3)(y+5)$
26. $(x-8)(x+8)$
27. $(c+9)(c+9)$
28. $(t-4)(t-4)$
29. $(3a+2)(2a+4)$
30. $(4y-2)(3y+5)$
31. $(2x+4)(5x-9)$
32. $(d+4)(d+7)$
33. $(c-9)(c+9)$
34. $(y+7)(y+7)$
35. $(a-2)(a-2)$
36. $(5a+1)(a-2)$
37. $(2x+1)(x-2)$
38. $(4y+3)(3y-9)$

39. $(t^2+3)(t^2+5)$
40. $(p-9)(p+2)$
41. $(r^2-3)(r^2-5)$
42. $(y^2+3)(y^2+3)$
43. $(b^3-11)(b^3-11)$
44. $(e^2+3)(e^2+3)$
45. $(4y+10)(5y-8)$
46. $(ab+2c)(ab+3c)$
47. $(6abc-d)(2abc+d)$
48. $(7x^2t+y)(2x^2t-5y)$
49. $(9x^2+3y)(9x^2+3y)$
50. $(8a-4y)(8a-7y)$
51. $(f+s)(f+s)$
52. $(f-s)(f-s)$

53. **Illustrative example.** What expression will represent the area of a square if each side is $x+6$ inches?

Solution: Evidently the area is $(x+6)(x+6)$, or $x^2+12x+36$. This is usually written, however, not as $(x+6)(x+6)$ but as $(x+6)^2$. The exponent, 2, shows that $x+6$ is used twice as a factor. Thus, $(3x+5)(3x+5)$ is usually written as $(3x+5)^2$. In the same way, $(3x+5)(3x+5)(3x+5)$ would be written as $(3x+5)^3$.

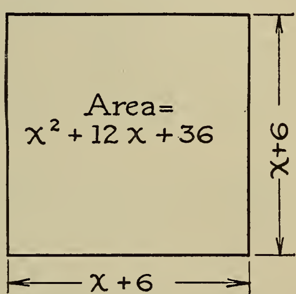


FIG. 136

54. $(x+5)^2$
55. $(3b+5)^2$
56. $(2y+7)^2$
57. $(4a-3)^2$
58. $(x+y)^2$
59. $(x-y)^2$
60. $(a+b)^2$
61. $(a-b)^2$
62. $(f+s)^2$
63. $(f-s)^2$
64. $(2x+3y)^2$
65. $(4b-2)^2$
66. $(a+b)(c+d)$

Section 125. Further practice in translating from algebraic symbols into word statements. You have had some

practice in translating from algebraic statements into word statements. For example, you translated the algebraic expression "$a + b$" into the word statement "the sum of two numbers," and the expression "xy" into "the product of two numbers." It is important to be able to translate into word statements some of the examples which you did in the last exercise.

EXERCISE 106

Write out the word statement which means the same thing as each of the following algebraic expressions:

1. $a - b$
2. a^2
3. x^3
4. $(2y)^3$
5. $c + d$
6. $a^2 + b^2$
7. $(a + b)^2$
8. $(a - b)^2$
9. $(a + b)(a - b)$
10. $r^3 + s^3$
11. $(f + s)^2 = f^2 + 2fs + s^2$
12. $(f - s)^2 = f^2 - 2fs + s^2$

Section 126. An important special product. One kind of multiplication occurs so frequently in later work that we should make a special study of it here. It results in a very common kind of product. It is the kind represented by Examples 11 and 12 in Exercise 106 and by Examples 54 to 65 in Exercise 105. Look back at these again.

Note that in each example **an expression** (either the sum or the difference of two numbers) **has been squared.** The **expression** which is squared **contains two terms**, and is generally **called a binomial.** The product which is obtained by squaring the two-term expression (binomial) always contains three terms and is called **a trinomial square.** In writing out word statements for Examples 11

and 12, Exercise 106, you probably obtained the following results:

$$(f+s)^2 = f^2 + 2fs + s^2.$$

The word statement is as follows:

The square of the sum of any two numbers equals the square of the first number, plus twice the product of the two numbers, plus the square of the second number.

$$(f-s)^2 = f^2 - 2fs + s^2.$$

For No. 12, the word statement is:

The square of the difference of any two numbers equals the square of the first number, minus twice the product of the two numbers, plus the square of the second number.

These statements are frequently used as rules for squaring the sum or difference of two numbers, *i.e.* for squaring binomials.

EXERCISE 107

Square the following binomials by the most economical method that you know:

1. $(x+y)^2$

2. $(a+b)^2$

3. $(c-d)^2$

4. $(x-y)^2$

5. $(2x+y)^2$

6. $(y+10)^2$

7. $(x-8)^2$

8. $(y^2-8)^2$

9. $(2x+5)^2$

10. $(3y-4)^2$

11. $(10a-2)^2$

12. $(5c-3)^2$

13. $(2x^2-3y)^2$

14. $(5c^2-4y^3)^2$

15. $(x+\frac{1}{2})^2$

16. $(y-\frac{1}{3})^2$

17. $(2y - \frac{1}{2})^2$

18. $(6b + \frac{1}{2})^2$

19. $(5c - \frac{1}{3})^2$

20. $(9a^2 - \frac{1}{6})^2$

21. $\left(x + \frac{a}{2}\right)^2$

22. $(y + \frac{1}{2}p)^2$

23. $\left(x - \frac{y}{2}\right)^2$

24. $\left(\frac{x}{2} - 4\right)^2$

25. $(5b + \frac{1}{4})^2$

26. $(3y - \frac{1}{4})^2$

27. In squaring each of these binomials, how many terms did you get?

28. How is the first term always obtained? the last term? the middle term?

29. If you knew the trinomial square, could you tell what binomial had been squared to give it? How?

30. State what binomial has been squared to give each of the following trinomial squares:

(*a*) $x^2 + 2xy + y^2$
(*b*) $y^2 + 6y + 9$
(*c*) $c^2 + 10c + 25$
(*d*) $a^2 - 12a + 36$
(*e*) $d^2 + 14df + 49f^2$
(*f*) $25x^2 + 30xy + 9y^2$
(*g*) $16b^2 - 80bc + 100c^2$
(*h*) $30y + 225y^2 + 1$

31. The following are the first and third terms of trinomial squares. Find the middle term for each.

(*a*) $y^2 + ? + x^2$
(*b*) $c^2 - ? + 25$
(*c*) $a^2 + ? + 16$
(*d*) $25x^2 + ? + 1$
(*e*) $16a^2 + ? + 9$
(*f*) $36y^4 + ? + 9$
(*g*) $y^2 + ? + \frac{1}{4}$
(*h*) $x^2 + ? + \frac{1}{9}$
(*i*) $a^2 - ? + \frac{1}{16}$
(*j*) $9x^2 + ? + 25y^2$
(*k*) $4a^2 + ? + \frac{1}{4}$
(*l*) $100x^2y^2 + ? + 25$
(*m*) $c^2a^2 + ? + x^2$
(*n*) $t^2 - ? + \frac{4}{9}$
(*o*) $x^2 - ? + \frac{4}{25}$
(*p*) $p^2 + ? + 49t^4$

32. The following are the first and middle terms of trinomial squares. Find the third term of each:

(*a*) $x^2 + 6x + ?$	(*j*) $16a^2 + 40a + ?$
(*b*) $y^2 - 8y + ?$	(*k*) $25c^2 - 80c + ?$
(*c*) $c^2 - 10y + ?$	(*l*) $x^2 + x + ?$
(*d*) $b^2 + 20b + ?$	(*m*) $y^2 + y + ?$
(*e*) $a^2b^2 - 12ab + ?$	(*n*) $a^2 - a + ?$
(*f*) $x^4 - 14x^2 + ?$	(*o*) $b^2 + \frac{1}{2}b + ?$
(*g*) $y^6 - 4y^3 + ?$	(*p*) $r^2 + \frac{1}{3}r + ?$
(*h*) $4x^2 + 12x + ?$	(*q*) $x^2 - \frac{1}{4}x + ?$
(*i*) $9y^2 + 30y + ?$	(*r*) $4t^2 + t + ?$

STANDARDIZED PRACTICE EXERCISE E (TIMED)

Practice on this exercise until you can reach the standard, 14 examples right in 4 minutes.

1. $(2x + 3y)^2$	**10.** $(4y + 7)(3y - 2)$.
2. $(y^2 - 5)^2$	**11.** $(6a + 5b)^2$ . . . . .
3. $(2a + 3)(2a - 3)$.	**12.** $(r^2 - 8)^2$ . . . .
4. $(2b^3 + 1)(2b^3 + 3)$	**13.** $(5c + 3)(5c - 3)$.
5. $(3x + 2)(2x - 5)$.	**14.** $(4y^4 + 1)(4y^4 + 2)$
6. $(5y + 4w)^2$	**15.** $(6t + 2)(2t - 3)$.
7. $(p^3 - 3)^2$	**16.** $(2a + 7c)^2$ . . . . .
8. $(4a + 5)(4a - 5)$.	**17.** $(x^4 - 7)^2$ . . . .
9. $(3s^2 + 4)(3s^2 + 5)$	**18.** $(3y + 4)(3y - 4)$.

Section 127. How to solve equations which involve products. The solution of a great many equations depends

upon your being able to find products like those you have just been finding. To illustrate, consider the problem:

Illustrative example. **The length of a rectangle is 6 inches more than, and the width is 2 inches less than, the sides of a square; the area of the rectangle exceeds the area of the square by 20 square inches. What are the dimensions of each?**

Solution: Let $s =$ the side of the square.

Translating into algebra, we have the equation:

$$(s+6)(s-2)=s^2+20.$$

Multiplying, or removing parentheses, gives

$$s^2+4s-12=s^2+20.$$

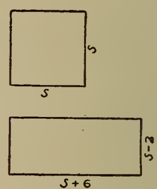
Subtracting s^2 from each side gives

$$4s-12=20.$$

$$\therefore s=8.$$

Thus, the sides of the rectangle are 14 and 6.

Check this result.



EXERCISE 108

PRACTICE IN SOLVING EQUATIONS WHICH INVOLVE PRODUCTS

Solve and *check* each of the following equations:

1. $(x+6)^2=x^2+96$
2. $(y+3)(y+6)-y^2=63$
3. $(2b+3)(b+4)=(b+1)(2b+10)$
4. One number is 5 larger than another; the square of the larger exceeds the square of the smaller by 55. Find each number.
5. The length of a rectangle is 8 inches more than, and its width is 3 inches less than, the sides of a square; the area of the rectangle exceeds the area of the square by 26 square inches. Find the dimensions of the rectangle.
6. $(y+5)(y-5)=(y-6)(y+2)$
7. $2(x-8)-3(x-4)=-5x$
8. $(b+4)^2-(b-2)^2=10$

9. $(x+3)^2-(x-1)^2=40$
10. $(y+5)^2-(y+4)^2=-1$
11. $(x-8)^2=(x-12)^2$
12. $2(f+3)^2=2(f-8)^2$
13. $3(x+6)(x+4)=(3x+1)(x+9)$

HOW TO FIND THE FACTORS OF AN ALGEBRAIC EXPRESSION

A. FINDING THE *COMMON* FACTOR

Section 128. Meaning of the word FACTOR. If you know that the area of a rectangle is 24 square inches, what might be its dimensions? You readily see here that the dimensions **might** be 4 inches and 6 inches; or 3 inches and 8 inches; or 12 inches and 2 inches, because the *product* of 4 and 6, or of 3 and 8, or of 12 and 2, is in each case 24. **This process of finding the numbers which, when multiplied together, give another, is called FACTORING.** The numbers you find are called the **FACTORS.** Thus, 4 and 6 are factors of 24; also 3 and 8, or 12 and 2.

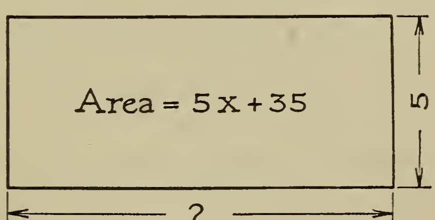


FIG. 137

The same reasoning is used in algebra as in arithmetic. For example, suppose the area of a rectangle is $5x+35$ square units. If the width is 5 units, what must the length

be? Similarly, if the area is $4x^2 + 28x$ and the width is $4x$, what is the length?

Section 129. What is a common factor? Now let us take a more difficult illustration. In the previous two cases, you knew both the *area* and one dimension, or the product and one of its factors. But in most *factoring problems* you do not know *any* of the factors. For example, what are the dimensions of the rectangle whose area is $7x + 21$, or, in other words, *what are the factors of* $7x + 21$? A study of the $7x$ and 21 shows that 7 is a factor of each, or is a **COMMON FACTOR** of both terms. What, then, must 7 be multiplied by to give $7x + 21$, or, what must be the length of this rectangle if its width is 7? Evidently, it must be $x + 3$. This problem should be written

$$7x + 21 = 7(x + 3).$$

A second illustration should make clear what is meant by factoring in algebra.

Find the factors of $ax + ay + aw$, or find the dimensions of a rectangle if its area is $ax + ay + aw$.

By observing each term, we see that a is a factor **common** to all the terms.

Dividing each term of the expression by a gives the other factor, $x + y + w$. Hence, $ax + ay + aw = a(x + y + w)$, and a is one factor and $x + y + w$ is the other one.

These illustrations are intended to make clear how to factor an expression in which there is a factor **common to all the terms.**

EXERCISE 109

Factor each of the following expressions:

1. $3x + 12$
2. $5a - 20$
3. $ax + bx$
4. $7a - 21$

5. $17x + 34$
6. $6a + 9$
7. $5 + 15a$
8. $9 + 6x^2$
9. $4y^2 + 12$
10. $8a + 12b$
11. $ab + ac + ah$
12. $5 + 10a + 15a^2$
13. $2x^2 - 4xy + 2y^2$
14. $2 - 8d^2$
15. $5x^2 - 5y^2$
16. $4a^2 + 8ab + 4b^2$
17. $x^2 + x$
18. $a^2 + 20a^3$
19. $x^2 + 5x^3$
20. $a + ab + a^2$
21. $x^2 - 5x^4$
22. $4a^3 - 12a^5$
23. $5x^2y - 5xy$
24. $7b^3 - 21b^2$
25. $12ab + 6a$
26. $x^2 + 2xy$
27. $2 - 20x$
28. $64x^2 - 24x^5$
29. $6ax^2 - 12ax^3$

Can you check the examples in this exercise? Check this one:

$$5x^2 - 15x^4 = 5x^2(1 - 3x).$$

B. THE CROSS-PRODUCT METHOD OF FACTORING

Section 130. In the previous section all the expressions which you factored had a *common factor*. But most expressions which you will need to factor are much more difficult than those; and they do not always contain a common factor.

Illustrative example. **Factor $2x^2 + 5x + 3$, or find the dimensions of a rectangle having this area.**

From what you learned about products, you can see that this expression was very likely made by multiplying two factors together. Also, you can see that the first terms of the two factors must be $2x$ and x. To help you to get the correct result, always **write the blank form of**

the two parentheses first, thus: (   )(   ). Then as you determine each term of each factor, you can write it in the appropriate place. Later you will doubtless be able to do all the work in your head and not have to write out the steps. **Second**, therefore, **write the first terms** in the blank form, thus:

$$2x^2 + 5x + 3 = (2x \quad\;)(x \quad\;).$$

Third, you have to find the **second term** of each factor. From your previous work in finding products, you know that the last term of the expression, $+3$, was obtained by multiplying together the second terms of the two factors. Then the second terms must be such that their product is $+3$. Obviously, they are 1 and 3 or 3 and 1. To tell whether the 3 or the 1 belongs in the first factor we have to **try** it, and test or check to see if the middle term will be correct $(+5x)$. Trying this out, we have:

$$2x^2 + 5x + 3 = (\overbrace{2x + \underbrace{1)(x}_{x} + 3}^{6x}).$$

Checking, — that is, multiplying the two factors together, — we see that this does *not* give the correct middle term, for $+6x$ and $+x$ are not $5x$. But, we might interchange the 1 and 3. Trying this, we get

$$2x^2 + 5x + 3 = (\overbrace{2x + \underbrace{3)(x}_{3x} + 1}^{2x}).$$

Multiplying these together shows that our factors are correct, for their product gives the original expression, $2x^2 + 5x + 3$.

Let us try another example:

Second illustrative example. Factor $5x^2 - 36x + 7$.
First write the blank form thus : (   )(   ).
Next we can tell at once that the first terms of our factors are $5x$ and x, giving

$$5x^2 - 36x + 7 = (5x \quad)(x \quad).$$

Examining the last term of the expression $+7$, we see that the second terms of the required factors must be 1 and 7 or 7 and 1. Trying out the 1 and 7 gives

$$5x^2 - 36x + 7 = (5x + 1)(x + 7).$$

But the check shows that the sum of the cross-products is $+36x$, whereas it should be $-36x$. This can be corrected by changing the sign of the second terms to -1 and -7, giving

$$(5x - 1)(x - 7).$$

The result of checking shows these to be the correct factors.

Section 131. The sum of the cross-products must equal the middle term. These two explanations have been given to show the importance of getting, as factors, expressions such that the sum of the cross-products will give the middle term of the expression to be factored. It is assumed that you can tell, at a glance, what the first terms **might** be, and what the second terms **might** be by looking at the first and last terms of the expressions which you want to factor.

For example, in factoring

$$6x^2 + 13x + 6,$$

the first terms **might** be $3x$ and $2x$, or $6x$ and x; the second terms **might** be 3 and 2, 2 and 3, or 6 and 1. But since the product of the factors must give the original expression, we can tell by trying these various possible combinations that the factors are

$$\overset{4x}{\overbrace{}}$$
$$\underbrace{(3x + \overbrace{2)(2x}^{4x} + 3)}_{9x}.$$

No other arrangement of numbers will give the correct middle term, $+ 13x$.

From this explanation you should be able to factor the expressions in Exercise 110. Don't be discouraged if you have to try more than once before you succeed. Difficult tasks often require many trials.

EXERCISE 110

FACTORING BY THE CROSS-PRODUCT METHOD

Check each example carefully.

(The parentheses are written here to suggest to you how to begin.)

1. $x^2 + 5x + 6 = (\quad)(\quad)$
2. $y^2 + 10y + 21 = (\quad)(\quad)$
3. $2x^2 + 7x + 5 = (2x + ?)(x + ?)$
4. $c^2 + 6c + 9 = (\quad)(\quad)$
5. $x^2 - 8x + 12 = (\quad)(\quad)$
6. $5y^2 + 16y + 3 = (5y + ?)(y + ?)$
7. $a^2 + 12a + 36 =$
8. $5x^2 + 8x + 3 =$
9. $x^2 - 2x - 24 =$
10. $m^2 - m - 20 =$
11. $2b^2 + 13b + 15 =$
12. $3x^2 - 13x + 4 =$
13. $t^2 - 5t - 40 =$
14. $10y^2 + 13y - 3 =$
15. $4x^2 + 20x + 25 =$
16. $a^2 + a - 72 =$
17. $y^2 - 16 =$
18. $b^2 - 25 =$
19. $15c^2 - 31c + 14 =$
20. $21b^2 - b - 2 =$

21. $5x^2 - 6x + 1$	**27.** $a^2 + 6a + 9$
22. $3x^2 + 4x + 1$	**28.** $y^2 - 8y + 16$
23. $2y^2 - y - 28$	**29.** $p^2 + 10$
24. $2a^2 + 7a + 3$	**30.** $c^2 + c - 30$
25. $x^2 - 11x + 24$	**31.** $3x^2 + 8x + 5$
26. $y^2 - 10y + 20$	**32.** $2x^2 - 5x + 3$

Section 132. Importance of finding the prime factors. Any algebraic expression that **cannot** be factored is **PRIME**. For example, $3x + 5$ is prime, because there are no integral expressions which can be multiplied together to produce it. But $9x + 6$ is *not prime* because it can be obtained by multiplying 3 and $x + 2$.

It is important that you should always find *prime factors*. To illustrate:

First illustrative example. Factor $3b^2 - 21b + 36$.
By inspection, we see that 3 is a common factor.
Removing it, we have

$$3(b^2 - 7b + 12).$$

Now, unless we remember that *prime* factors should be found, we are likely to leave the example in this incomplete form. The $b^2 - 7b + 12$ can be factored further, however, giving

$$(b - 3)(b - 4).$$

Thus, the original example should be factored as follows:

$$\begin{aligned} 3b^2 - 21b + 36 &= 3(b^2 - 7b + 12) \\ &= 3(b - 4)(b - 3). \end{aligned}$$

Second illustrative example. Another illustration will make clear the importance of finding *prime factors*. Suppose we wish to factor $2x^2 - 50$. As in the previous example, we *always first look for a common factor*. This gives

$$2(x^2 - 25).$$

Now, again, we are apt to leave the example in this incomplete form, not remembering to see if we can further factor $x^2 - 25$. We see, however, that we can. The complete solution is:

$$\begin{aligned} 2x^2 - 50 &= 2(x^2 - 25) \\ &= 2(x + 5)(x - 5). \end{aligned}$$

These explanations are given to help you keep in mind that in all factoring work **there are two absolutely essential steps**; namely,

1. **LOOK FOR A COMMON FACTOR.**
2. **FIND PRIME FACTORS; *I.E.* FACTOR COMPLETELY.**

EXERCISE 111

PRACTICE IN FACTORING COMPLETELY

1. $2a^2 + 14a + 24$
2. $5y^2 - 45$
3. $st^2 - st - 20s$
4. $7a^2 - 14a - 105$
5. $3x^2 + 12x + 45$
6. $x^2 - 6x + 9$
7. $6t^2 - 15t^3$
8. $2 - 128t^2$
9. $ab^2 - ab - 72a$
10. $6x^2 + 13x + 6$
11. $3a^2 + a - 2$
12. $2a^2 - 5a + 3$
13. $7b^2 - 17b - 12$
14. $q^2 - 12q - 28$
15. $2x^2 - 14x + 24$
16. $y^4 - 6y^2 - 16$
17. $49c^2 + 70c + 25$
18. $5a^2 - 80$
19. $x^3 - x^2$
20. $6x^2 - 18x^3$
21. $3y^2 - 3y - 36$
22. $12x^2 + 37x - 10$
23. $10x + 25 + x^2$
24. $y^2 + 10$
25. $x^2 + 36$
26. $2a^2 - 8$
27. $3b^3 + 27$
28. $3x^2 - 12x - 180$
29. $2x^2 + 10x - 168$
30. $x^2 - x - 110$
31. $2y^2 - y - 1$
32. $6a^2 - 4a - 2$

33. $3x^2 + 4x + 1$
34. $20x^2 + 70x + 60$
35. $2b^2 - b - 3$
36. $2a^2 + 18$
37. $5y - 15y^2$
38. $a^2 - b^2$
39. $3y^2 - y - 10$
40. $2x^2 + 3x - 9$
41. $6y^2 + y - 15$
42. $8x^2 - 2$
43. $18t^2 - 50$
44. $9x^2 + 17x - 2$
45. $6a^2 - a - 12$
46. $7y^2 - 9y - 10$
47. $2x^2 - 36x + 64$
48. $y^2 - 3y - 4$
49. $p^2 - 16$
50. $p^2 + 16$
51. $2y^2 - 2y - 24$
52. $b^2 - 24 + 2b$
53. $-30 - x + x^2$
54. $4x^2 + 20x + 25$
55. $c^3 + 2c^2 + c$
56. $p^4 - 81$

STANDARDIZED PRACTICE EXERCISE F (TIMED)

Practice on this exercise until you can reach the standard, 12 examples right in 4 minutes. Factor each expression.

1. $6x^2 - 18x^3$	10. $4p^4 - 3p$ . . . .
2. $2y^2 + 10y + 12$ . .	11. $2y - 16y^3$ . . . . . .
3. $5 - 30d^2$	12. $5x^2 + 15x + 10$ . .
4. $r^4 + 7r^3 + 12r^2$.	13. $6 - 24p$ . . . .
5. $3d^3 - 11d^4$	14. $b^4 + 6b^3 + 5b^2$ . . .
6. $7x^3 - 14x^5$	15. $5r^6 - 8r^9$ . . . . . .
7. $3b^2 + 18b + 24$ . .	16. $3a^2 - 15a^5$ . . . . .
8. $9 - 18q$	17. $4w^2 + 16w + 12$.
9. $x^5 + 3x^4 + 2x^3$ . .	18. $7 - 21q$ . . . .

STANDARDIZED PRACTICE EXERCISE G (TIMED)

Practice on these examples until you can reach the standard, 12 examples right in 6 minutes. Factor each expression.

1. $3x^2 - 6x - 24$ . .	10. $25d^2 + 30dc + 9c^2$
2. $9x^4 - 25y^2$.	11. $5a^2 - 20a - 60$ . .
3. $p^4 - p^2 - 20$	12. $z^4 - 81w^6$ . . . . . .
4. $p^2 + 5p + 10$ . . .	13. $y^8 - y^4 - 30$ . . . . .
5. $9x^2 + 12xy + 4y^2$	14. $d^2 + 7d + 11$ . . . .
6. $2y^2 - 6y - 20$ . .	15. $49p^2 + 28pq + 4q^2$
7. $16r^6 - 49y$	16. $3b^2 - 15b - 150$.
8. $x^6 - x^3 - 12$	17. $4n^{10} - 121x^8$ . . . .
9. $c^2 + 3c + 8$	18. $x^2 - x - 56$ . . . . .

REVIEW EXERCISE 112

1. What is the area of a square formed by adding 4 ft. to the sides of a square x ft. long?
2. What does $(x - 4)(x + 6)$ represent, if x represents the side of a square?
3. A rectangular field $5y$ rods long has a perimeter of $24y$ rods. What expression will represent the area of the field in square rods? in acres?
4. If the quotient is represented by q, the divisor by d, and the remainder by r, what will represent the dividend?
5. If a park is w rods wide and 1 rod long, how many miles would you walk in going around it n times?
6. How do you divide a product of several factors

by a number? For example, in dividing $12 \cdot 3 \cdot 6$ by 2, would you divide each factor by 2?

7. How do you multiply a product of several factors by a number? Give an illustration.

8. The product of four factors is 60. Three of them are 2, 3, and 5. Find the fourth factor.

9. How much do you increase the area of a square whose side is x, if you increase its side 4 units?

10. Make a detailed summary for this chapter.

11. Solve for x, explaining each step:

$$-4x + 6 = 2x - 18.$$

12. In what way is *factoring* like *division?* How is it like multiplication?

13. Solve: $\begin{cases} 6y - x = 7 + 4y, \\ 5x + 8y = 1. \end{cases}$

14. Make up five examples for the class to factor, and then give them to the class to work.

15. How many terms do you get when you square the *sum* of *two numbers*, *e.g.* $(2x + 3y)^2$? when you square the *difference* of *two numbers*, *e.g.* $(4a - 3b)^2$?

16. Evaluate $(a + b)^3$ if $a = -3$ and $b = +1$.

17. Does $(a + b)^2 = a^2 + b^2$? Show by using 4 for a and 5 for b.

18. A tree stands on a bluff on the opposite side of a river from the observer. Its foot is at an elevation of 45° and its top at 60°. Which has the greater height, the bluff or the tree? What measurement would you have to make to find the height of the tree? the width of the river?

19. Translate: $a^2 - b^2 = (a + b)(a - b)$.

CHAPTER XIV

THE USE OF FRACTIONS WITH LETTERS

Section 133. What an algebraic fraction means. In arithmetic a fraction was used to represent one or more of the equal parts of some unit. For example, the fraction $\frac{4}{5}$ meant 4 of the 5 equal parts into which something had been divided. In algebra, however, a fraction has a more general meaning. It is thought of as a quotient, or an indicated division. Thus, the fraction $\frac{a}{b}$ means "the quotient of a and b," and is read "a divided by b"; or "a over b." Examples of algebraic fractions are:

$$\frac{D^2N}{2 \cdot 5}, \qquad \frac{3}{T}, \qquad \frac{a+b}{ab}.$$

Section 134. Numerator and denominator: the terms of a fraction. Just as in arithmetic, we have the terms **numerator** and **denominator** of fractions. The numerator is the dividend, and the denominator is the divisor.

Section 135. How to change fractions into other equivalent fractions. In arithmetic we frequently change fractions into **equivalents**, for example, $\frac{8}{10}$ may be changed to $\frac{4}{5}$, or $\frac{6}{8}$ may be changed to $\frac{3}{4}$ by dividing the numerator and denominator of each fraction by 2. In the same way $\frac{ab}{ac}$ may be reduced to **the equivalent**, $\frac{b}{c}$, by dividing the numerator and denominator by a. Similarly $\frac{(a+b)}{(a+b)(a-b)}$ is reduced to $\frac{1}{a-b}$ by dividing both numerator and denominator by $(a+b)$.

EXERCISE 113—ORAL WORK

Tell what has been done to the first fraction in each of the following examples to give the corresponding equivalent fraction:

1. $\frac{12}{15}=\frac{4}{5}$

2. $\frac{10}{18}=\frac{5}{9}$

3. $\frac{a}{b}=\frac{ac}{bc}$

4. $\frac{xy}{ay}=\frac{x}{a}$

5. $\frac{x^2y}{xw}=\frac{xy}{w}$

6. $\frac{pq}{rq}=\frac{p}{r}$

7. $\frac{a^2bc}{ab}=ac$

8. $\frac{ab^2}{a^2bc}=\frac{b}{ac}$

9. $\frac{2\cdot 3}{2\cdot 5}=\frac{3}{5}$

10. $\frac{5\cdot 4\cdot 2}{7\cdot 5\cdot 2}=\frac{4}{7}$

11. $\frac{a\cdot b\cdot c}{a\cdot c\cdot d}=\frac{b}{d}$

12. $\frac{x}{y}=\frac{abx}{aby}$

13. $\frac{x+1}{4}=\frac{2x+2}{8}$

14. $\frac{y+2}{5}=\frac{3(y+2)}{15}$

15. $\frac{a}{a+b}=\frac{5a}{5(a+b)}$

TWO IMPORTANT PRINCIPLES IN HANDLING FRACTIONS

Section 136. The examples in Exercise 113 illustrate two very important principles in fractions.

I. To multiply both numerator and denominator of a fraction by the same expression does not change the value of the fraction.

II. To divide both numerator and denominator of a fraction by the same expression does not change the value of the expression.

These principles are used in changing fractions into equivalent fractions. The next exercise will give practice in doing this. State what principle is used in each change.

EXERCISE 114

PRACTICE IN CHANGING FRACTIONS INTO EQUIVALENT FRACTIONS

Supply the missing term in each of the following, and tell what fundamental principle has been used in changing the first fraction into the second fraction:

1. $\frac{3}{5}=\frac{?}{15}$

2. $\frac{4}{3}=\frac{20}{?}$

3. $\frac{ab}{bc} = \frac{a}{?}$

4. $\frac{x}{x+5} = \frac{?}{3x+15}$

5. $\frac{4b}{a+3} = \frac{?}{5a+15}$

6. $\frac{x+y}{x-y} = \frac{?}{(x-y)(x+y)}$

7. $\frac{a-2}{a+6} = \frac{?}{(a+6)(a+3)}$

8. $\frac{x+3}{(x+5)(x+3)} = \frac{?}{x+5}$

9. $\frac{b}{a+b} = \frac{2b}{?}$

10. $\frac{a+b}{a-b} = \frac{?}{a^2-2ab+b^2}$

11. $\frac{x+2}{x+3} = \frac{?}{x^2+5x+6}$

12. $\frac{b}{x+5} = \frac{?}{x^2+5x}$

13. $\frac{(a+b)(a-b)}{(a-b)} = \frac{a+b}{?}$

14. $\frac{(x+2)}{(x+2)(x-2)} = \frac{?}{x-2}$

15. $\frac{p+5}{(p+5)(p+2)} = \frac{1}{?}$

16. $\frac{x^2+5x+6}{x^2+7x+12} = \frac{x+2}{?}$

17. $\frac{a^2-6a+9}{a^2-8a+15} = \frac{?}{a-5}$

Section 137. Reducing fractions to lowest terms. You have been taught in arithmetic always to reduce fractions to their lowest terms. It is equally important to reduce algebraic fractions to lowest terms. Just as in arithmetic, a fraction is in its lowest terms when it has no factor common to both numerator and denominator. Hence, to reduce fractions to lowest terms, it is necessary to divide each numerator and denominator by all factors common to both.

EXERCISE 115

PRACTICE IN REDUCING FRACTIONS TO THEIR LOWEST TERMS

1. Reduce to lowest terms: $\frac{4}{20}, \frac{3}{15}, \frac{14}{36}, \frac{a}{ab}, \frac{bc}{cd}, \frac{x^3y}{xy}$.

2. Reduce to lowest terms: $\frac{ax}{a^2y}$, $\frac{2xy}{6xw}$, $\frac{6a^2b}{9ab^2}$, $\frac{2x^3y^2}{6x^2y^2}$.

Reduce each of the following to lowest terms:

3. $\frac{4a^3bc^2}{12ab^2c^2}$ **5.** $\frac{2 \cdot wy^2w^5}{12yw^4}$ **7.** $\frac{2(a+b)}{6(a+b)}$

4. $\frac{4 \cdot 5 \cdot x}{2 \cdot 5 \cdot y}$ **6.** $\frac{120a^2bc^4}{2 \cdot 3 . 4a^2c^4}$ **8.** $\frac{30(x+y)}{6(x+2y)}$

Factor numerator and denominator of each of the following, and then reduce to lowest terms:

9. $\frac{a^2-b^2}{a^2+2ab+b^2}$ **13.** $\frac{9b^2-c^2}{9b^2-6bc+c^2}$

10. $\frac{x^2-4}{x^2+5x+6}$ **14.** $\frac{t^2-4}{2t+4}$

11. $\frac{b^2+7b+12}{b^2+5b+6}$ **15.** $\frac{(x+2)^2}{x^2-4}$

12. $\frac{6a^2-30}{6x+12y}$ **16.** $\frac{a^2-ab}{a^3-a^2b}$

17. $\frac{ax-5a}{x^2-25}$

HOW TO ADD OR SUBTRACT ALGEBRAIC FRACTIONS

Section 138. Use of the most convenient denominator. We learned in arithmetic that fractions may not be added or subtracted unless they have the same denominator. Only like fractions may be combined by addition or subtraction. Therefore we had to change fractions into **equivalent** fractions which had the same denominator.

For example: to add $\frac{2}{3}$ and $\frac{5}{8}$ we must change each fraction to 24ths.

The same reasoning holds with fractions with letters.

We must make the fractions have the same denominator.

For example, to add $\frac{a}{b}+\frac{c}{d}$, we must change each fraction into an equivalent fraction in which bd is the denominator, just as 24 is the denominator in the foregoing example.

Illustrative example. Showing how to change fractions into equivalent fractions which have the same denominators.

Change the fractions $\frac{a}{bd}$ and $\frac{b}{cd}$ into equivalent fractions having the same denominator.
The most convenient denominator must contain b, d, and c, and be of lowest possible degree. Consequently it is bdc.
To change the first denominator, bd, to bdc, it is necessary to multiply the numerator and denominator by c. (Why multiply the numerator also by c?) This gives $\frac{ac}{bdc}$ as a fraction equivalent to $\frac{a}{bd}$. In the same way the second denominator, cd, must be changed to bdc. Multiplying both terms of the fraction $\frac{b}{cd}$ by b, gives $\frac{b^2}{bdc}$ as an equivalent fraction. Thus we have changed the two given fractions into equivalent fractions which have the same denominator.
Summarizing: $\frac{a}{bd}=\frac{ac}{bdc}$ and $\frac{b}{cd}=\frac{b^2}{bdc}$.

Section 139. How to find the most convenient denominator. It has been pointed out that fractions must have the **same denominator** before they can be combined. In the following exercise you are to decide what the **most convenient denominator** is in each example, and then to make the fractions in each example have that denominator.

EXERCISE 116

MAKE THE FRACTIONS IN EACH OF THESE EXAMPLES HAVE THE SAME DENOMINATOR

1. $\frac{a}{b}, \frac{c}{b}$
2. $\frac{x}{y}, \frac{3w}{y}$
3. $\frac{a}{b}, \frac{c}{d}$
4. $\frac{ab}{c}, \frac{cd}{a}$
5. $\frac{a}{b}, \frac{1}{4}$
6. $\frac{x}{2y}, \frac{3}{y}$
7. $\frac{a+b}{5x}, \frac{a-b}{x}$
8. $\frac{b+3}{y}, \frac{b}{2y}$
9. $\frac{a}{bc}, \frac{d}{ab}$
10. $\frac{x}{wy}, \frac{y}{wx}$
11. $\frac{a^2b}{c^2d}, \frac{ab^2}{c^2d}$
12. $\frac{2c}{5y}, \frac{c^2}{10y}$
13. $\frac{5}{a+b}, \frac{4}{a+b}$
14. $\frac{x}{x+2}, \frac{x}{x+3}$
15. $\frac{5}{b-2}, \frac{3}{b-3}$
16. $\frac{x+2}{x+3}, \frac{x+4}{x+5}$

17. From the experience you have had in solving the previous examples, you will be able to answer the following questions concerning how to find the most convenient denominator.

(*a*) What is the most convenient denominator for two fractions which have the same denominator?
(*b*) What is the most convenient denominator for two fractions which have unlike denominators with no factors common to each? Give some illustrations from the preceding exercise.
(*c*) What is the most convenient denominator for two or more fractions which have unlike denominators, but which have a factor common to both denominators? Illustrate from the preceding exercise.

18. Add $\frac{3}{5}$ and $\frac{4}{5}$.
19. Add $\frac{a}{b}$ and $\frac{c}{b}$.
20. Add $\frac{4}{5}$ and $\frac{7}{9}$.
21. Add $\frac{a}{b}$ and $\frac{b}{d}$.

We have learned how to change fractions into equivalent fractions which have the same denominator. Now we are ready to get practice in adding and subtracting fractions.

EXERCISE 117

ORAL PRACTICE IN ADDING AND SUBTRACTING FRACTIONS

1. $\frac{5}{2}+\frac{6}{2}$; $\frac{2}{5}+\frac{3}{5}$; $\frac{a}{2}+\frac{3a}{2}$; $\frac{x}{3}+\frac{x}{3}$; $\frac{y}{4}+\frac{5y}{4}$; $\frac{2}{a}+\frac{3}{a}$.

2. $\frac{6}{5}-\frac{2}{5}$; $\frac{2a}{5}-\frac{a}{5}$; $\frac{3x}{4}-\frac{x}{4}$; $\frac{2y}{9}+\frac{5y}{9}$; $\frac{a+b}{5}+\frac{a-b}{5}$.

3. $\frac{a}{b}+\frac{c}{b}$; $\frac{x}{y}-\frac{w}{y}$; $\frac{2a}{c}-\frac{b}{c}$; $\frac{5b}{3}-\frac{c}{3}$; $\frac{2p}{5}+\frac{3q}{5}$.

4. $\frac{2}{3}+\frac{1}{7}$; $\frac{2}{5}+\frac{3}{10}$; $\frac{4}{6}-\frac{1}{14}$; $\frac{6}{4}+\frac{1}{8}$; $\frac{7}{3}-\frac{1}{6}$; $\frac{2}{9}+\frac{4}{3}$.

5. $\frac{a}{2}+\frac{3a}{4}$; $\frac{4c}{5}+\frac{c}{10}$; $\frac{y}{6}-\frac{2y}{3}$; $\frac{3x}{5}+\frac{9x}{10}$; $\frac{b}{2}-\frac{7b}{4}$.

6. $\frac{a+b}{6}+\frac{a+b}{3}$; $\frac{a-b}{5}+\frac{a+b}{10}$; $\frac{x-y}{4}+\frac{x-y}{8}$;

$$\frac{2a+b}{3}+\frac{a+b}{6}.$$

7. $\frac{a}{b}+\frac{c}{d}$; $\frac{1}{a}+\frac{1}{b}$; $\frac{1}{x}-\frac{1}{y}$; $\frac{x}{y}-\frac{w}{z}$; $\frac{1}{b}-\frac{1}{c}$.

Section 140. Steps in adding or subtracting fractions. In the addition or subtraction of fractions it is most economical to follow a definite procedure. A study of the examples in the previous exercise suggests the following steps or procedure for combining fractions.

First step: If the fractions have the same denominators, write the algebraic sum of the numerators over the common denominator.

What examples in the previous exercise suggest this rule?

Second step: If the fractions have unlike denominators, change the fractions into equivalent fractions which have the

same denominator, and write the algebraic sum of the numerators over the common denominator.

What examples suggest this rule?

Third step: The resulting fraction should be reduced to lowest terms.

EXERCISE 118

PRACTICE IN ADDING AND SUBTRACTING FRACTIONS

Do as many as you can orally:

1. Add: $\frac{c}{d}$ and $\frac{b}{d}$; $\frac{x^2}{y}$ and $\frac{5}{y}$; $\frac{c}{a^2}$ and $\frac{d}{a^2}$; $\frac{1}{r}$ and $\frac{p}{r}$.

2. $\frac{a+b}{2}$ and $\frac{a-b}{2}$; $\frac{x+y}{a}$ and $\frac{x-y}{a}$; $\frac{c+d}{m}$ and $\frac{c-d}{m}$.

3. $\frac{a}{b}$ and $\frac{c}{d}$; $\frac{x}{y}$ and $\frac{w}{z}$; $\frac{ab}{x}$ and $\frac{bc}{y}$; $\frac{a}{bc}$ and $\frac{d}{ac}$.

4. $\frac{a^2+b^2}{x+y}$ and $\frac{a^2-b^2}{x+y}$.

5. $\frac{c^2+d^2}{x-y}$ and $\frac{c^2-d^2}{x-y}$.

6. $\frac{a^2-10}{x+y}$ and $\frac{a^2+12}{x+y}$.

7. $\frac{x+y}{3}$, $\frac{x-y}{3}$, and $\frac{y}{3}$.

Add and check each of the following:

8. $\frac{x+y}{6}+\frac{x-y}{3}$.

9. $\frac{a+b}{5}+\frac{a-b}{4}$.

10. $\frac{a^2-b^2}{4}+\frac{a^2+2ab+b^2}{8}$.

11. $\frac{x}{3}+\frac{2x}{5}-\frac{2x}{15}$.

12. $\frac{a}{b}+\frac{c}{d}+\frac{e}{f}$.

13. $\frac{x-y}{2}-\frac{x-y}{6}$.

14. Add $\frac{1}{a}+\frac{1}{b}$. Think of the result which you obtain as a formula for adding any two such fractions. Write this formula as a word rule. Apply this formula to the addition of

$$\frac{1}{4} \text{ and } \frac{1}{5}; \quad \frac{1}{3} \text{ and } \frac{1}{4}; \quad \frac{1}{5} \text{ and } \frac{1}{6};$$

$$\frac{1}{3} \text{ and } \frac{1}{7}; \quad \frac{1}{8} \text{ and } \frac{1}{5}; \quad \frac{1}{x}+\frac{1}{y}.$$

15. Show that $\frac{1}{a}-\frac{1}{b}=\frac{b-a}{ab}$. Think of this as a formula for finding the difference between two such fractions. Write the rule in words. Apply the formula to

$$\frac{1}{3}-\frac{1}{5}; \quad \frac{1}{5}-\frac{1}{4}; \quad \frac{1}{2}-\frac{1}{5}; \quad \frac{1}{8}-\frac{1}{9}; \quad \frac{1}{3}-\frac{1}{7}; \quad \frac{1}{x}-\frac{1}{y}.$$

16. Show that $\frac{a}{b}+\frac{x}{y}=\frac{ay+bx}{by}$. Think of this as a formula for adding any two such fractions. State the rule in words. Apply this formula to

$$\frac{3}{4}+\frac{5}{7}; \quad \frac{2}{5}+\frac{3}{4}; \quad \frac{1}{6}+\frac{2}{5}; \quad \frac{8}{3}+\frac{5}{2};$$

$$\frac{2}{9}+\frac{3}{4}; \quad \frac{a}{b}+\frac{c}{d}; \quad \frac{c^2}{d^2}+\frac{a^2}{b^2}.$$

17. Show that $\frac{a}{b}-\frac{x}{y}=\frac{ay-bx}{by}$. Think of this as a formula, and apply to:

$$\frac{3}{4}-\frac{2}{5}; \quad \frac{7}{3}-\frac{2}{6}; \quad \frac{8}{7}-\frac{2}{9}; \quad \frac{1}{4}-\frac{3}{5}; \quad \frac{2}{9}-\frac{3}{4}; \quad \frac{b^2}{c}-\frac{x^2}{y}.$$

18. $\frac{a}{5}-\frac{2a}{3}+\frac{7a}{15}$

19. $\frac{2b}{7}-\frac{b}{3}+\frac{5b}{21}$

20. $\frac{4}{x}+\frac{5}{x^2}-\frac{6}{x^3}$

21. $\frac{3}{a^3}-\frac{3}{2a^2}+\frac{1}{a}$

22. $\frac{4}{y}+\frac{3}{5y}+\frac{2}{y^2}$

23. $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$

24. $\dfrac{x+2}{x-2}+\dfrac{5x}{(x+2)(x-2)}$

25. $\dfrac{5}{b+2}+\dfrac{6}{b-3}$

26. $\dfrac{3}{x+4}+\dfrac{5}{x+3}$

27. $\dfrac{4}{x+2}+\dfrac{2}{x+3}$

28. $\dfrac{4}{b-5}+\dfrac{3}{b+2}$

29. $\dfrac{x+2}{5}-\dfrac{x-6}{3}$

30. $\dfrac{2b+1}{4}-\dfrac{3b-4}{7}$

31. $\dfrac{a-b}{c}-\dfrac{c-a}{b}$

32. $\dfrac{5}{x-2}-\dfrac{3}{x+6}$

33. $\dfrac{8}{y+5}-\dfrac{5}{y+8}$

34. $\dfrac{a+2}{a+3}-\dfrac{a-1}{a+7}$

35. $\dfrac{y-1}{y+4}-\dfrac{y-2}{y+5}$

36. $\dfrac{5}{x^2-4}+\dfrac{6}{x+2}$

37. $\dfrac{x}{x^2-9}+\dfrac{3}{x-3}$

38. $\dfrac{4}{x^2+5x+6}+\dfrac{x}{x+2}$

39. $\dfrac{x+7}{x^2+8x+15}+\dfrac{x+5}{x+3}$

40. $3+\frac{1}{7}$

41. $5+\frac{2}{9}$

42. $1+\frac{4}{7}$

43. $1+\dfrac{6}{x}$

44. $1+\dfrac{8}{3y}$

45. $a+\frac{2}{5}$

46. $1+\dfrac{b}{c}$

47. $1-\dfrac{a}{b}$

48. $1-\dfrac{x^2}{y^2}$

49. $1+\dfrac{a^2}{b^2}$

50. $b+\dfrac{a}{c}$

51. $x+\dfrac{1}{x}$

52. $y^2-\dfrac{1}{y}$

53. $c-\dfrac{a}{b}$

54. $\dfrac{1}{t}+\dfrac{1}{T}$

55. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$

56. $\dfrac{1}{f}+\dfrac{1}{F}$

57. $\dfrac{1}{x+2}-\dfrac{1}{x+3}$

58. $\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}$

59. $1-\dfrac{x-y}{x+y}$

60. If a certain pipe can fill a tank in 12 minutes, what part of the tank can it fill in 1 minute? What part could it fill in 1 minute if it requires t minutes to fill the tank?

61. A tank can be filled by one pipe in t minutes, and by another pipe in T minutes. What part of the tank can be filled in 1 minute if both pipes are open? Express this as one fraction.

62. If $A=\frac{1}{x}$ and $B=\frac{1}{2x}$, what does $A+B$ equal? What does $A-B$ equal?

63. Find $A+B$ if $A=\frac{1}{x+5}$ and $B=\frac{1}{x-4}$.

HOW TO MULTIPLY ALGEBRAIC FRACTIONS

Section 141. Multiplication of fractions. You have seen that addition and subtraction in algebra are performed exactly the same as in arithmetic. Multiplication of fractions, also, is performed in exactly the same manner. The following exercise will illustrate.

ORAL EXERCISE 119

1. $5\times\frac{2}{3}$; $6\times\frac{3}{5}$; $7\times\frac{2}{3}$; $5\times\frac{3}{8}$; $8\times\frac{2}{5}$; $12\times\frac{3}{7}$.
2. $a\cdot\frac{2}{3}$; $b\cdot\frac{3}{5}$; $2a\cdot\frac{1}{3}$; $3y\cdot\frac{2}{5}$; $5x\cdot\frac{3}{4}$; $\frac{2}{3}\cdot 4y$.
3. $\frac{1}{3}\times\frac{4}{5}$; $\frac{2}{5}\times\frac{4}{3}$; $\frac{3}{7}\times\frac{2}{5}$; $\frac{1}{4}\times\frac{5}{2}$; $\frac{4}{5}\times\frac{3}{7}$.
4. $\frac{a}{b}\times\frac{c}{d}$; $\frac{x}{y}\times\frac{a}{b}$; $\frac{a}{c}\times\frac{c}{b}$; $\frac{2b}{3c}\times\frac{3a}{4d}$.
5. $\frac{4}{5}\times\frac{10}{2}$; $\frac{6}{4}\times\frac{8}{12}$; $\frac{9}{8}\times\frac{4}{3}$; $\frac{12}{5}\times\frac{10}{8}$.
6. $\frac{ab}{c}\times\frac{c}{a}$; $\frac{bc}{ad}\times\frac{d}{c}$; $\frac{x}{y}\cdot\frac{w}{y}$.

Section 142. Steps in multiplication of fractions. In solving the examples in the previous exercise, you made use of the following steps or rules:

To multiply a fraction by an integral expression, multiply the numerator by the integral expression and write the result over the denominator. For example:

$$7 \times \frac{3}{5} = \frac{7 \cdot 3}{5} = \frac{21}{5};\ a \times \frac{b}{c} = \frac{a \cdot b}{c} = \frac{ab}{c}.$$

To multiply a fraction by a fraction, multiply the numerators together for a new numerator, and the denominators together for a new denominator. For example:

$$\frac{3}{5} \times \frac{7}{8} = \frac{3 \cdot 7}{5 \cdot 8} = \frac{21}{40};\ \frac{a}{b} \times \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}.$$

To reduce the result to lowest terms, divide numerator and denominator by any factor that is common to both. For example:

$$\frac{5}{7} \times \frac{14}{9} = \frac{5}{\cancel{7}} \times \frac{\overset{2}{\cancel{14}}}{9} = \frac{10}{9},\ \frac{a}{b} \times \frac{b}{c} = \frac{a \cdot \cancel{b}}{\cancel{b} \cdot c} = \frac{a}{c}.$$

EXERCISE 120. MULTIPLICATION OF FRACTIONS

1. $y \times \frac{x}{y}$
2. $a \times \frac{c}{a}$
3. $bc \times \frac{b}{c}$
4. $5 \times \frac{y^2}{5}$
5. $4a \times \frac{b}{4c}$
6. $bc^2 \times \frac{a}{bc}$
7. $x^2y \cdot \frac{1}{xy}$
8. $(a+b) \cdot \frac{1}{a+b}$
9. $mn \cdot \frac{1}{m}$
10. $5 \cdot 3 \cdot \frac{1}{3 \cdot 4}$
11. $a^2b^2 \cdot \frac{1}{abc}$
12. $\frac{a}{b} \cdot \frac{b}{c}$
13. $\frac{xy}{ab} \cdot \frac{a^3b^4}{x}$
14. $\frac{m^4n}{xy^2} \cdot \frac{y^3}{m^4}$
15. $\frac{2 \cdot 3^2}{5 \cdot 7} \cdot \frac{35}{2^2 \cdot 3}$
16. $\frac{x+2}{x+3} \cdot \frac{x+3}{x+1}$

17. $(x+2)(x-2)\cdot\dfrac{1}{x+2}$ **18.** $(y+3)(y-3)\cdot\dfrac{1}{y+3}$

19. $\dfrac{2x^2}{3y}\cdot\dfrac{6y^2}{5x}$. Check by letting $x=1$ and $y=3$.

20. Multiply $2\cdot 3\cdot 4$ by 5.

21. Multiply $5^2\cdot 3\cdot 4$ by 2.

22. Multiply a^2bc by b.

23. Multiply xy^2w by x^2.

24. State the rule for multiplying a *product* by a number.

25. Give a formula for the product of any two fractions; of any three fractions.

26. How do you affect the value of a fraction by multiplying its numerator by a number? by dividing its denominator by a given number? by multiplying both its numerator and denominator by a given number? by dividing both numerator and denominator by a given number?

27. Is the value of a fraction changed by adding the same expression to its numerator and denominator? Illustrate. Is its value changed by subtracting the same expression from its numerator and denominator? Illustrate.

28. Do you change the value of a fraction by squaring both numerator and denominator? Illustrate.

29. Tell what has been done to the first fraction in each of the following, to give the second fraction, and then tell whether or not the value has been changed:

(a) $\frac{a}{b}, \frac{ac}{bc}$ (c) $\frac{x}{y}, \frac{x^2}{y^2}$ (e) $\frac{a}{b}, \frac{a-x}{b-x}$

(b) $\frac{a}{b}, \frac{a+1}{b+1}$ (d) $\frac{xy}{yw}, \frac{y}{w}$ (f) $\frac{bc}{ad}, \frac{b}{a}$

30. Summarize all the operations that may be performed upon a fraction without changing its value.

31. $\frac{x+2}{x-3}\cdot(x+3)(x-3)$

32. $\frac{b+6}{b-1}\cdot(b+2)(b-1)$

33. $\frac{c^2-3}{c+2}\cdot(c+2)(c^2+3)$

34. $\frac{x-7}{x-4}\ (x-4)(x-1)$

35. $(y+6)(y-4)\cdot\frac{y-5}{y-4}$

36. $(b+c)(b-c)\cdot\frac{b+c}{b-c}$

37. $(x+y)(x-y)\cdot\frac{1}{x-y}$

38. $(x^2+5x+6)\cdot\frac{1}{x+3}$

39. Multiply a^2+8a+6 by $\frac{a+3}{a+2}$. *First, factor* a^2+8a+6.

40. y^2+6y+9 by $\frac{y+2}{y+3}$

41. $x^2-3x-10$ by $\frac{x-2}{x-5}$

42. c^2+6c+5 by $\frac{c-1}{c+5}$

43. r^2-9 by $\frac{r+4}{r+3}$

44. p^2-6p by $\frac{4}{p}$

45. b^2-8b by $\frac{b+9}{b}$

HOW TO DIVIDE FRACTIONS

Section 143. Division of fractions. In arithmetic, you learned how to divide one fraction by another, *i.e.* to invert the divisor, and proceed as in multiplication of fractions.

For example:

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}; \quad \frac{3}{4} \div 5 = \frac{3}{4} \times \frac{1}{5} = \frac{3}{20}.$$

In the same way, in algebraic division, we invert the divisor and multiply.

EXERCISE 121

Do as many orally as you can.

1. Divide: $\frac{b}{3}$ by $\frac{b}{6}$; $\frac{b}{6}$ by $\frac{b}{3}$; $\frac{3}{b}$ by $\frac{6}{b}$; $\frac{6}{b}$ by $\frac{3}{b}$.

2. Divide: $\frac{a}{b}$ by $\frac{c}{d}$; $\frac{x}{y}$ by $\frac{5}{y}$; $\frac{bc}{ad}$ by $\frac{b}{a}$.

3. Divide: $\frac{x+y}{2}$ by $\frac{x-y}{2}$; $\frac{a+b}{x}$ by $\frac{a-b}{x}$; $\frac{x}{a+b}$ by $\frac{x}{a-b}$.

4. $\frac{ab}{xy} \div \frac{bc}{xy}$

5. $\frac{c^2d}{ab} \div \frac{c^3d^2}{a^2b}$

6. $\frac{a}{b} \div \frac{b}{c}$

7. $\frac{1}{x} \div \frac{x}{y}$

8. $\frac{1}{a} \div \frac{1}{A}$

9. $\frac{x}{y} \div \frac{1}{y}$

10. Divide $\frac{8x^2y}{9ab^2}$ by $\frac{2xy}{3b^2}$. Check by letting $x = 1$, $y = 2$, $a = 3$, and $b = 4$.

11. $\frac{10x^3}{14y^2} \div \frac{15x^2}{21y^5}$

12. $\frac{12bc^2}{5a} \div \frac{18b^2c^2}{15a^3}$

13. $\frac{4x^2y}{51b} \div \frac{2xy}{17b^3}$

14. $\frac{(2ab)^2}{25c^2} \div \frac{18a^3b}{100c^3}$

15. $\frac{(x+y)(x-y)}{(a+b)(a-b)} \div \frac{x+y}{a-b}$

16. $\frac{x^2-y^2}{a^2-b^2} \div \frac{x-y}{a+b}$

17. $\frac{2\frac{1}{2}}{5} \div \frac{3\frac{2}{3}}{4}$

18. $\frac{(4ab)^2}{(2y)^3} \div \frac{8a^2}{6x^2y^3}$

19. $\dfrac{x^2+6x+8}{a^2+4a+4} \div \dfrac{x+2}{a+2}$ **20.** $\dfrac{x^2-25}{x^2-9} \div \dfrac{x+5}{x-3}$

21. Tell by what the first expression in each of the following must be divided to give the second expression.

(*a*) $-3x,\ x$
(*b*) $bcy,\ y$
(*c*) $\frac{1}{2}gt,\ t$
(*d*) $\dfrac{by}{2},\ y$
(*e*) $abxy,\ y$
(*f*) $lwh,\ h$
(*g*) $\dfrac{ab}{2},\ b$
(*h*) $\pi R^2,\ R^2$
(*i*) $c^2x,\ x$
(*j*) $NTx,\ x$
(*k*) $\frac{5}{9}Ft,\ t$
(*l*) $m(a+b)y,\ y$

SUMMARY OF CHAPTER XIV

The following points should be mastered by the study of this chapter:

1. **An algebraic fraction is a quotient, or an indicated division. Its terms are numbers, with letters.**

2. **The terms of a fraction may be multiplied or divided by the same expression without changing the value of the fraction.**

3. **Fractions are reduced to their lowest terms by dividing both terms by all factors which are common to both. Hence, to reduce algebraic fractions, it is usually necessary to factor both terms.**

4. **Only like fractions can be added or subtracted. If we wish to add or subtract unlike fractions, we must change them into equivalent fractions which have the same denominator.**

5. **The most convenient denominator to which fractions must be changed (in order to add or subtract**

them) is the expression of lowest degree which will contain each of the separate denominators.

6. Multiplication and division of algebraic fractions are performed exactly as in arithmetic.

REVIEW EXERCISE 122

1. Think of

$$\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab} \text{ and } \frac{1}{a}-\frac{1}{b}=\frac{b-a}{ab}$$

as formulas for the addition and subtraction of a special kind of fractions. Apply these formulas to the following:

(*a*) $\frac{1}{5}+\frac{1}{2}$
(*b*) $\frac{1}{4}+\frac{1}{3}$
(*c*) $\frac{1}{6}+\frac{1}{7}$
(*d*) $\frac{1}{2}+\frac{1}{3}$
(*e*) $\frac{1}{4}-\frac{1}{5}$
(*f*) $\frac{1}{6}-\frac{1}{10}$
(*g*) $\frac{1}{4}-\frac{1}{7}$
(*h*) $\frac{1}{7}-\frac{1}{8}$
(*i*) $\frac{1}{x^2}+\frac{1}{y^2}$
(*j*) $\frac{1}{b^2}-\frac{1}{c^2}$
(*k*) $\frac{1}{ab}-\frac{1}{cd}$
(*l*) $\frac{1}{a+b}+\frac{1}{c+d}$

2. Think of

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \text{ and } \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{ad}$$

as formulas for the addition and subtraction of fractions. Apply these formulas to the following:

(*a*) $\frac{2}{3}+\frac{5}{7}$
(*b*) $\frac{3}{4}+\frac{5}{7}$
(*c*) $\frac{2}{5}+\frac{3}{4}$
(*d*) $\frac{9}{2}+\frac{1}{5}$
(*e*) $\frac{5}{4}-\frac{3}{7}$
(*f*) $\frac{6}{11}-\frac{2}{9}$
(*g*) $\frac{4}{3}-\frac{1}{2}$
(*h*) $\frac{6}{4}-\frac{7}{8}$
(*i*) $\frac{a}{bc}+\frac{1}{x}$
(*j*) $\frac{xy}{w}-\frac{1}{x}$
(*k*) $\frac{lw}{2}-\frac{3}{x}$
(*l*) $\frac{a^2}{b^2}-\frac{c}{ad}$

3. Change to a single fraction: $\frac{1}{A}+\frac{1}{B}+\frac{1}{C}$.

4. Reduce to lowest terms: $\frac{2^2 \cdot 3 \cdot 5^3}{2^3 \cdot 3^2 \cdot 5}$; $\frac{a^3b^4c^2}{a^5b^3c}$; $\frac{2^2 \times 5^3}{4 \times 25 \times 23}$.

5. $A = \frac{1}{x} - \frac{1}{y}$ and $B = \frac{1}{x-y}$. Which expression, A or B, has the greater numerical value, if $x = 5$ and $y = 3$?

6. Does $\frac{a^2 + b^2}{a + b} = a + b$? Check.

7. If $A = \frac{4}{3}$ and $B = \frac{5}{2}$, what is the ratio of A to B?

8. Why should the chapter on Products and Factors come before the chapter on Fractions?

9. Solve with two unknowns; two thirds of a certain number, increased by $\frac{3}{4}$ of another number, gives 25; the sum of the two numbers is 35. Find each number.

10. The value of a certain fraction is $\frac{2}{3}$; the sum of the numerator and denominator is 25. Find the fraction.

11. $\begin{cases} 2x - y = 8 \\ x - 2 = y + 1 \end{cases}$ 12. $\begin{cases} x + y = b - a \\ 2x - y = -5a - b \end{cases}$

13. The sum of two numbers, x and y, is a, and their difference is b. Find each number.

CHAPTER XV

LITERAL AND FRACTIONAL EQUATIONS

Section 144. A new kind of equation: literal equations. Thus far in the equations which you have solved, the only letter used was the letter used to represent the unknown. All other numbers were **arithmetical** or ordinary **numbers. Now, we shall study equations** which, in some cases, will not contain any ordinary numbers. That is, the **known numbers may be represented by letters** instead of being expressed in ordinary numbers. For example, consider the equation, or formula, for the volume of a rectangular box,

$$lwh = V.$$

There are cases in which we wish to restate this equation in a new form so as to give the value of l, that is, the *length*. In such a case l would be the unknown number. In other cases we might wish to restate the formula so as to find the value of w, that is, the *width*; then w would be regarded as the unknown number. If we consider l as the unknown in the equation, then w, h, and V are regarded as *known* numbers and are treated just as if they were arithmetical numbers.

Two illustrations will make this clear.

Illustrative example.

Solve for l the equation, $lwh = V$.

Solution: $lwh = V$.

Dividing each side of the equation by wh, gives

$$l = \frac{V}{wh}.$$

By what would you have divided to solve for w? for h? for V?

Second illustrative example.

Solve for x the equation,

$$ax + bx = c.$$

Solution: $ax + bx = c.$

(1) Factoring the left side, to get x by itself gives

$$x(a + b) = c.$$

(2) Dividing each side by $a + b$ gives

$$x = \frac{c}{a + b}.$$

The next exercise will give practice in solving easy equations with letters.

EXERCISE 123 — ORAL WORK

Solve each of these equations:

Consider the letters toward the end of the alphabet, either x, y, or z for example, as the unknowns, and those toward the beginning as the knowns, unless otherwise directed.

1. $y - b = 0$
2. $x - b = 2b$
3. $w + 4a = 7a$
4. $x - 3c = -5c$
5. $y + 5a = -4a$
6. $x - b = c$
7. $y = 2a = b$
8. $w - 3c = 2b$
9. $2x = 4b - 6c$
10. $5x - 3c = 17c$
11. $ay = a$
12. $by = bc$
13. $cx = c^2d$
14. $4ax = 12ac$
15. $3by = 9bc + 6b$
16. $3x - 6a = 9b$
17. $2y - 4b = 2c$
18. $ax = a^2 + 2ab$
19. $5ay + 4ac = 14ac$
20. $bx - bc = ab$
21. $x(a + b) = a + b$
22. $y(c - d) = (c - d)(c - d)$
23. $w(a + b) = (a + b)(a - b)$
24. $x(a + b - c) = 4a(a + b - c)$
25. $y(a + b)(a - b) = b(a - b)(a + b)$

EXERCISE 124

Solve for the unknown in each of the following, and check:

Illustrative example.

Solve	$4y - b = y + 5b.$
Transposing,	$4y - y = 5b + b.$
Collecting,	$3y = 6b.$
Dividing by 3,	$y = 2b.$
Checking,	$4 \cdot 2b - b = 2b - 5b.$
	$8b - b = 2b + 5b.$
	$7b = 7b.$

1. $3x - c = x + 9c$
2. $4y + b = y + 4b$
3. $x - a = a + b$
4. $4by + 2b = 14b$
5. $ax + ac = ab$
6. $cx - c^2 = bc$
7. $6b^2y + b^2 = 7b^2$
8. $-ax + ab = -ac$
9. $bc - by = ab$
10. $cx - bc = c^2$
11. $b - 2x = 5b - 3x$
12. $y + 2(a - b) = 4a - b$
13. $2(y - b) = 6bc$
14. $-(x - c) = -3c$
15. $-b(x - 3) = 7b$
16. $2bx - 3ab = 6ab - bx$
17. $\frac{1}{2}bx = 4b - \frac{1}{2}bx$
18. $x(a + b) - 2a = 2b$
19. $y(c - a) = 2(c - a)$
20. $2ay + ac = ab$
21. **Illustrative example.** $ax - ac = bc - bx$

(1) Transposing, $ax + bx = bc + ac.$

(2) Factoring the left side, to get x by itself gives

$$x(a + b) = ac + bc.$$

(3) Dividing by $a + b$, gives

$$x = \frac{ac + bc}{a + b} = \frac{c(a + b)}{a + b} = c.$$

Checking: $a \cdot c + b \cdot c = ac + bc.$

22. $ay - by = ac - bc$
23. $3cx + 2bx = 6c + 4b$
24. $ax - bx = ac - bc$
25. $a - p = 4a + 4$
26. $by + y = 3b + 3$
27. $2ay - by = (2a - b)(a - b)$
28. $4ax - x = 16a^2 - 1$
29. $by - 2y = b^2 - 5b + 6$
30. $ax + 5x = a^2 + 8a + 15$
31. $bx + 4 = x + 4b.$

Section 145. Equations containing fractions with literal denominators. We have already solved many fractional equations in which the denominators were ordinary arithmetical numbers. (For example, in Chapter II, such examples as $\frac{2}{3}p + \frac{3}{4}p = p + 5$.) Now we shall take up the solution of equations which contain **fractions with literal denominators.**

Recall how you solved equations such as

$$\frac{1}{2}x - \frac{x}{5} = 3.$$

Note that you found the most convenient multiplier, 10, and then got rid of all fractions in the equation, by multiplying each term by 10.

In the same way, you solve such equations as

$$\frac{x}{a} + \frac{x}{b} = a + b.$$

That is, you **find the most convenient multiplier to use in order to get rid of fractions.** Why is it ab in this equation? Then **the equation is of the same form as those in the previous section.**

Illustrative example.

Solve $$\frac{x}{a} + \frac{x}{b} = a + b.$$

(1) Multiplying each term of the equation by ab gives

$$bx + ax = a^2b + ab^2.$$

(2) Factoring the left member to get x by itself gives

$$x(b + a) = a^2b + ab^2.$$

(3) Dividing by $b + a$ gives

$$x = \frac{a^2b + ab^2}{b + a} = \frac{ab(a + b)}{b + a} = ab.$$

Check: Let the student check the example.

EXERCISE 125

Do Examples 1–15 orally.

1. $\frac{4}{x} = 2$
2. $\frac{3}{y} = 1$
3. $\frac{15}{2x} = 5$
4. $\frac{x}{a} = b$
5. $\frac{y}{2b} = 3$
6. $\frac{x}{bc} = 2$
7. $\frac{2x}{3b} = 4$
8. $\frac{c}{x} = b$
9. $\frac{b}{y} = a$
10. $\frac{a}{x} = c$
11. $\frac{4a}{3x} = 2$
12. $\frac{3x}{2b} = 6b$
13. $\frac{x}{a} + \frac{1}{2} = 5$
14. $\frac{y}{2} + \frac{1}{b} = 0$
15. $\frac{x}{a} + \frac{1}{a} = \frac{5}{a}$

Each of the equations in Examples 16 to 27 is a practical formula. These formulas are used commonly in various kinds of work. It is often necessary to be able to solve them for any letter. The rest of this exercise, therefore, will give you practice in solving such practical formulas.

16. Solve $A = bh$ for b; for h.
17. Solve $V = lwh$ for l; for w.
18. Solve $C = \frac{E}{R}$ for E; for R.
19. Solve $d = rt$ for r; for t.
20. Solve $i = prt$ for p; for t.
21. Solve $P = \frac{D^2N}{2 \cdot 5}$ for N.
22. If $\frac{a}{b} = c$, what does b equal?
23. If $a(x + c) = bc$, what does x equal?
24. Solve $A = \frac{bh}{2}$ for b; for h.

25. If $A = \frac{h(a+b)}{2}$, what does h equal? What does a equal?

26. Given the formula $a = p(1 + rt)$, solve for p, and then for r.

TIMED PRACTICE EXERCISE H

Practice under time to see how many examples you can do in 5 minutes. The very rapid pupils in 100 cities did about 20; the average pupils did about 12.

1. $i = prt$ Solve for r	12. $Q = \frac{\pi r^2 v}{h}$ Solve for v
2. $s = \frac{W}{L}$ Solve for L	13. $P = ahw$ Solve for h
3. $R = \frac{EI}{M}$ Solve for E	14. $c = \frac{E}{R}$ Solve for R
4. $C = \frac{Ka - b}{a}$ Solve for K	15. $E = \frac{PL}{K}$ Solve for P
5. $M = \frac{bh^3}{3}$ Solve for b	16. $L = \frac{Mt - g}{t}$ Solve for M
6. $r = \frac{v^2 pL}{a}$ Solve for p	17. $I = \frac{bd^3}{3}$ Solve for b
7. $V = LWh$ Solve for W	18. $E^2 = \frac{JWhr}{t}$ Solve for h
8. $v = \frac{S}{T}$ Solve for T	19. $v = \pi L r^2$ Solve for L
9. $s = \frac{ah}{r}$ Solve for a	20. $A = \frac{F}{M}$ Solve for M
10. $R = \frac{WL - x}{L}$ Solve for W	21. $M = \frac{SI}{T}$ Solve for S
11. $v = \frac{bh^3}{3}$ Solve for b	22. $f = \frac{gm - t}{m}$ Solve for g

Section 146. Equations containing fractions with binomial denominators. In the previous fractional equations, no denominator contained more than a single term. There will be occasions in your later work when you will need to be able to solve fractional equations in which the denominators contain expressions with two letters, *i.e.* binomial denominators. For example, suppose you were to solve the formula

$$C = \frac{E}{R + r} \text{ for } R;$$

it would be necessary to get rid of a binomial denominator, $R + r$. Therefore, we shall next learn how to solve such equations. Let us illustrate with the equation

$$\frac{x}{x+5} = \frac{1}{3}.$$

No new principle is required to get rid of fractions in this equation. The only difficulty is that involved in selecting a convenient multiplier, *i.e.* in finding the lowest common multiple of the denominators. In this example, we must get rid of the denominator $x + 5$, and of the denominator 3. Therefore we must multiply each term of the equation by $3(x + 5)$. Why?

Illustrative example.

$$\textbf{Solve } \frac{x}{x+5} = \frac{1}{3}.$$

(1) Multiplying by $3(x + 5)$ gives

$$3\cancel{(x+5)} \cdot \frac{x}{\cancel{x+5}} = \cancel{3}(x+5) \cdot \frac{1}{\cancel{3}}.$$

(2) Reducing gives

$$3x = x + 5$$

or

$$x = 2.5.$$

Check: $\dfrac{2.5}{2.5 + 5} = \dfrac{1}{3}$; or $\dfrac{2.5}{7.5} = \dfrac{1}{3}$; or $\dfrac{1}{3} = \dfrac{1}{3}$.

EXERCISE 126

PRACTICE IN SOLVING EQUATIONS WHICH CONTAIN FRACTIONS WITH BINOMIAL DENOMINATORS

1. $\dfrac{x}{x+5}=\dfrac{1}{2}$

2. $\dfrac{y}{y+2}=\dfrac{3}{2}$

3. $\dfrac{x-2}{x+3}=\dfrac{3}{8}$

4. $\dfrac{b-7}{b+2}=\dfrac{1}{4}$

5. $\dfrac{5}{x+3}=\dfrac{2}{x}$

6. $\dfrac{10}{y+7}=\dfrac{4}{y}$

7. $\dfrac{x}{x-3}=\dfrac{6}{3}$

8. $\dfrac{2}{y+4}=\dfrac{5}{y}$

9. $\dfrac{-3}{y+5}=\dfrac{1}{1}$

10. $\dfrac{x-1}{x-2}=1\frac{1}{2}$

11. $\dfrac{4}{x}=\dfrac{1}{x-3}$

12. $\dfrac{6}{5x}=\dfrac{-2}{x-3}$

Second illustrative example.

Solve $$\frac{x-4}{x+3}=\frac{x-16}{x+5}.$$

(1) Multiplying each side by $(x+3)(x+5)$ gives

$$\cancel{(x+3)}(x+5)\cdot\frac{x-4}{\cancel{x+3}}=(x+3)\cancel{(x+5)}\cdot\frac{x-16}{\cancel{x+5}}.$$

(2) Reducing gives $(x+5)(x-4)=(x+3)(x-16)$.

(3) Multiplying, $x^2+x-20=x^2-13x-48.$

(4) Transposing, $x^2+x-x^2+13x=20-48.$

(5) Collecting, $14x=-28,$

or $x=-2.$

Check: $\dfrac{-2-4}{-2+3}=\dfrac{-2-16}{-2+5}$, or $\dfrac{-6}{1}=\dfrac{-18}{3}$ or $-6=-6.$

13. $\dfrac{y+3}{y-4}=\dfrac{y+9}{y-5}$

14. $\dfrac{y-1}{y+1}=\dfrac{y+3}{y+10}$

15. $\dfrac{b-3}{b+5}=\dfrac{b-2}{b+2}$

16. $\dfrac{x+1}{x+4}=\dfrac{x-4}{x+2}$

17. $\dfrac{5}{y-1}=\dfrac{6}{1}$

18. $\dfrac{4}{x}+\dfrac{4}{x+2}=0$

19. $\dfrac{2(x-3)}{5}-\dfrac{3(4-x)}{2}=\dfrac{59}{5}.$

20. $\dfrac{1}{x+a}+\dfrac{1}{x-a}=\dfrac{4a}{(x-a)(x-a)}.$

21. $\dfrac{2}{x-3}+\dfrac{4}{x-3}=\dfrac{3}{x^2-9}.$

22. $\dfrac{1}{x}+\dfrac{x}{x+a}=1.$

23. Solve $C=\dfrac{E}{R+r}$ for R (a formula used in electricity).

24. Solve for l and n: $s=\dfrac{n(a+l)}{2}.$

25. Solve $C=\frac{5}{9}(F-32)$ for F (a formula used in changing thermometer readings from one scale (Fahrenheit) to another scale (Centigrade)).

26. Solve the formula $\dfrac{1}{q}-\dfrac{1}{p}=\dfrac{1}{f}$ for f; for p. (This is a formula used in studying lenses.)

27. If $A=\dfrac{1}{x+5}$ and $B=\dfrac{1}{x-5}$, find $A+B$ and $A-B$.

28. Solve $R=\dfrac{C}{N}+1$ for N.

29. Solve for A: $\dfrac{1}{A}=\dfrac{1}{B}+\dfrac{1}{C}.$

STANDARDIZED PRACTICE EXERCISE I (TIMED)

Practice on this exercise until you can reach the standard, 9 examples right in 10 minutes.

1. $\frac{2y-5}{3}-2-\frac{y-2}{5}=\frac{2y-3}{15}$ . . . .
2. $\frac{3}{x-8}-\frac{4}{x-10}=0$ . . . . .
3. $\frac{s+3}{s-8}=\frac{s+5}{s-2}$ . . . . . .
4. $\frac{b+4}{b-7}-\frac{b+1}{b+5}=\frac{b+6}{b^2-2b-35}$ . . . . .
5. $\frac{3y-2}{4}-3-\frac{2y-1}{3}=\frac{1-3y}{12}$ . . . .
6. $\frac{5}{x-2}-\frac{3}{x-6}=0$ . . .
7. $\frac{t+2}{t-7}=\frac{t+3}{t-4}$ . . . . .
8. $\frac{c+5}{c-8}-\frac{c+2}{c+6}=\frac{7c+3}{c^2-2c-48}$ . . . . .
9. $\frac{2w-1}{3}-1-\frac{3w-2}{7}=\frac{w+3}{21}$ . . . . .
10. $\frac{4}{x-3}-\frac{2}{x-9}=0$ . . . .
11. $\frac{r+5}{r-6}=\frac{r+4}{r-1}$ . . . . .
12. $\frac{y+2}{y-7}-\frac{y+4}{y+10}=\frac{9y+7}{y^2+3y-70}$ . . . . .

THE CONSTRUCTION OF GENERAL FORMULAS

Section 147. Now that we have learned how to solve literal equations we can construct formulas for a great many new types of problems. Let us illustrate and solve a few of these types.

Illustrative example. *A.* **PARTICULAR PROBLEM; that is, problem with arithmetical numbers. The sum of two numbers is 20, and the larger is 9 times the smaller. Find the numbers.**

Let $x =$ smaller no.

Then $9x =$ larger no.

$$x + 9x = 20.$$

Solving, $x = 2$ and $9x = 18$.

B. **GENERAL PROBLEM; that is, problem in which the numbers are all literal. The sum of two numbers is** S, **and the larger is** m **times the smaller. Find the numbers.**

Let $x =$ smaller no.

Then $mx =$ larger no.

Solving,

$$mx + x = s,$$

$$x(m+1) = s,$$

or

$$x = \frac{s}{m+1}.$$

Therefore,

$$mx = \frac{ms}{m+1}.$$

Note that **the particular problem is a special case of the general problem**; *i.e.* when $s = 20$ and $m = 9$. Thus, the expressions $\frac{s}{m+1}$ and $\frac{ms}{m+1}$ may be regarded as general expressions or formulas, for finding two numbers when their sum and ratio are known. In the following exercise, you will first solve a particular problem, with arithmetical numbers, and then solve a general problem which represents all the special problems of that type.

What are the numbers if $s = 30$ and $m = 4$? If $s = 100$ and $m = 9$?

EXERCISE 127

PRACTICE IN CONSTRUCTING GENERAL FORMULAS

1. (*a*) Particular problem: The sum of two numbers is 16; the larger is 4 more than the smaller. Find each number.

(*b*) General problem: The sum of two numbers is S; the larger is a more than the smaller. Solve for the two numbers, *i.e.* make a formula for finding each number in this type of problem. Evaluate when $S = 40$ and a is 10.

2. (*a*) The sum of two numbers is 32, and their difference is 8. Find each number.

(*b*) The sum of two numbers is S, and their difference is d. Find each number.

(*c*) Read the formulas you obtain in (*b*) as rules for finding the numbers.

(*d*) Evaluate when $S = 52$ and $d = 10$.

3. (*a*) The sum of two numbers is 9; 10 times the smaller equals 5 times the larger. Find the numbers.

(*b*) The sum of two numbers is S, and m times the smaller equals n times the larger. Find each number.

(*c*) Read the formulas in (*b*) as rules for finding the numbers.

(*d*) Evaluate when $S = 12$, $m = 6$, and $n = 2$.

4. (*a*) A rectangle is 10 ft. longer than it is wide; its perimeter is 72 ft. Find its length and width.

(*b*) A rectangle is b feet longer than it is wide; its perimeter is p feet. Find its length and width.

(*c*) Read the results in (*b*) as rules for solving any problem of this type.

(*d*) Make up a particular problem and solve it by the use of these formulas.

5. (*a*) The sum of two numbers is 14; three times their sum is equal to 21 times their difference. Find each number.

(*b*) The sum of two numbers is S; m times their sum is equal to n times their difference. Find each number.

(*c*) Read the formulas in (*b*) as rules for solving problems of this type.

(*d*) Make up a particular problem which you can solve by these formulas.

(*e*) In problems of this type, can $m = n$? Can m be greater than n? Can it be smaller than n?

6. (*a*) Separate 20 into two parts, such that the quotient of the larger by the smaller shall be $\frac{3}{2}$. Use x for the smaller.

(*b*) Separate n into two parts, such that the quotient of the larger by the smaller shall be $\frac{a}{b}$.

(*c*) Read the formulas which you obtained in (*b*) as rules for solving problems of this type.

(*d*) Make a particular problem which belongs to type.

(*e*) Can $a = b$? Can a be smaller than b? Why?

7. (*a*) Think of some number; multiply it by 3; add 9 to the result; multiply the last result by 2; divide the last result by 6; subtract

the number thought of in the beginning; the result must be 3. Can you tell why?

(*b*) Think of some number, as n; multiply it by a; then add $3a$; then multiply by 2; then divide by $2a$; then subtract the number thought of in the beginning. Show that the result must be a. Can you make up other problems similar to this?

EXERCISE 128

WORK AND RATE PROBLEMS

1. **Illustrative problem.** One contractor, Mr. A., has the facilities for building a house in 15 days; another contractor, Mr. B., has the facilities for building it in 20 days. The owner was very anxious to have the house completed at once; so he had both contractors work. How long would it take them, working together, to build the house?

A can do $\frac{1}{15}$ of the work in 1 day.

B can do $\frac{1}{20}$ of the work in 1 day.

Both together can do $\frac{1}{15}+\frac{1}{20}$ of the work in 1 day.

Let $x =$ no. of days required for both to do the work.

Then $\frac{1}{x} =$ what both can do in 1 day.

This gives the equation $\frac{1}{15}+\frac{1}{20}=\frac{1}{x}$.

Multiplying by $60x$, $4x + 3x = 60$,

or $x = 8\frac{4}{7}$ days for both to build house.

2. Contractor A can do a certain piece of work in 10 days; contractor B can do the same job in 8 days. How long would it take both working together to do the work?

3. The morning issue of a paper can be printed on one press in 5 hours, and on another press in 4

hours. How long would be required if both presses were used?

4. A painter can paint a house in 12 days; his apprentice would require 20 days. How long would both together require?

5. A tank can be filled by either of three pipes. Pipe A can fill it in 20 minutes; pipe B can fill it in 25 minutes, and pipe C can fill it in 50 minutes. How much time would be required if all three pipes were opened at the same time?

6. Solve for x: $\frac{1}{4}-\frac{1}{5}=\frac{1}{x}$, and $\frac{x}{4}-\frac{x}{5}=1$.

7. A can do a job in a days which B can do in b days. Both working together can do the work in how many days?

8. A works twice as fast as B. How long will it take both together to do what B alone can do in 10 days?

9. A can do a piece of work in a days, B in b days, and C in c days. In how many days can all working together do it?

PROBLEMS BASED UPON FRACTIONS

EXERCISE 129

1. **Illustrative problem.** **The numerator of a certain fraction is 5 smaller than its denominator; if 1 be added to the numerator, the value of the resulting fraction becomes $\frac{1}{2}$. Find the fraction.**

Let $\qquad d =$ **the denominator.**

Then $\qquad d-5=$ **the numerator,**

and $\qquad \frac{d-5}{d}=$ **the fraction.**

By the condition of the problem

$$\frac{d-5+1}{d}=\frac{1}{2}$$

or $$\frac{d-4}{d}=\frac{1}{2}.$$

Solving, $2d-8=d.$

$d=8$, the denominator.

$d-5=3$, the numerator.

$\frac{3}{8}=$ fraction.

Check: $$\frac{3+1}{8}=\frac{1}{2}; \text{ or } \frac{4}{8}=\frac{1}{2}.$$

2. The numerator of a certain fraction is 4 smaller than the denominator; if 1 be added to the numerator, the value of the resulting fraction becomes $\frac{2}{3}$. Find the fraction.

3. The denominator of a fraction exceeds its numerator by 3; if the numerator is decreased by 3, the value of the resulting fraction becomes $\frac{1}{4}$. Find the fraction.

4. The numerator of a certain fraction exceeds its denominator by 2. If the numerator be decreased by 1 and the denominator increased by 1, the value of the resulting fraction becomes 1. Find the fraction.

5. The denominator of a certain fraction is twice as large as the numerator. Find the value of the fraction. What is its value if the denominator is three times as large as the numerator?

6. The value of a certain fraction is $\frac{3}{4}$; if the numerator and denominator each be increased by 2, the value of the resulting fraction will be $\frac{4}{5}$. Find the fraction. (Use two unknowns, n and d.)

7. The value of a certain fraction is $\frac{4}{5}$; if the numerator be increased by 1 and the denominator decreased by 4, the value of the resulting fraction will be $\frac{3}{2}$. Find the fraction.

8. Make up a problem similar to one in this list. Your teacher will have the class solve it.

SUMMARY OF CHAPTER XV

1. **A literal equation is one in which some or all of the numbers are expressed by letters.**

2. **By transposing, all terms which contain the unknown letter are placed on one side of the equation; all other terms on the other side.**

3. **If more than one term contains the unknown, we must then separate the unknown factors from the knowns, by factoring. This gives the coefficient of the unknown, by which each side of the equation must be divided.**

4. **Literal equations may contain binomial denominators. If so, we get rid of fractions by multiplying each term in the equation by an expression which is the least multiple common to all the denominators.**

REVIEW EXERCISE 131

1. B weighs $\frac{2}{3}$ as much as A. The distance between them on a teeter board is 24 feet. How far is A from the fulcrum if they balance each other?

2. A can do a piece of work in 4 days, B in $6\frac{1}{2}$ days. How many will they require, working together?

3. An auto tourist made a trip of 120 miles, at a certain rate; on the return trip he increased his rate 5 miles per hour, and required 4 hours less time. Find his rate going.

4. What number must be subtracted from the denominator of the fraction $\frac{11}{20}$ to make the value of the fraction $\frac{11}{5}$ larger?

5. An athlete can run two and one half times as fast as he can walk. Find his speed in yards per second if he can run 100 yards in 12 less seconds than he can walk it.

6. A football player started toward his goal with the ball, for a touchdown. An opposing player was 10 yards behind him, but could run 2 yards a second faster. The first player had gone 30 yards when he was overtaken by the second. Find the rate of each, assuming that they started at the same time.

7. The sum of two angles is 140°, two thirds of the smaller equals one half of the larger. Find each.

8. A common number trick is: Think of a number; add 7; double the result; subtract 8; tell me your answer, and I will tell you the number you thought of. Can you explain why this is so?

9. A train running at the rate of r miles per hour required t hours to make a trip with n stops of s minutes each. What distance did it travel?

10. $(x+1)(x+2)-(x-3)(x-4)=0$. Find x and check.

11. A man receives w dollars each week in the year; his expenses average l dollars per week.

He takes a two weeks' vacation, during which his expenses are increased 25 dollars per week. How much will he save in a year?

12. Factor: (*a*) $30x^2 + 61x + 30$.

(*b*) $6x^2 - 13x + 6$.

13. A boy left a certain place, riding his motorcycle at the rate of 12 miles per hour; two hours later a second boy started to overtake him, riding at the rate of 16 miles per hour. In how many hours will he overtake him? Solve graphically.

14. If A should give B $6, they would have equal amounts, but if B should give A $1, A would have three times as much as B. How much has each?

15. Two unknown weights balance when placed on a teeter board, 8 and 10 feet from the fulcrum. If their positions are reversed, 54 pounds must be added to the lesser weight to make a balance. What are the weights?

16. A grocer has two kinds of sugar, one worth 10 ¢ a pound, the other worth 12 ¢. How many pounds of each must he use in a mixture of 50 pounds worth $5.60?

17. The admission price to a moving picture show was 10 ¢ for adults and 5 ¢ for children. One evening there were 420 admissions and receipts of $36. How many adults and how many children attended the show?

18. Solve both algebraically and graphically:

$$\begin{cases} 3x - y = 10, \\ 6x = 2y + 4. \end{cases}$$

CHAPTER XVI

HOW TO SHOW THE WAY IN WHICH ONE VARYING QUANTITY DEPENDS UPON ANOTHER

Section 148. Quantities that change together. We have already seen that there are many illustrations of quantities that change together. The amount of money paid out for rent at $30 per month **changes with, or depends upon,** the number of months; the time required to walk a certain distance, say 10 miles, **changes as, or depends upon,** the number of miles one walks per hour. In other words, there are **varying quantities** which are so **related** that **a change in the value of one of them causes a change in the value of the other.**

We could not continue our study of these varying quantities in Chapter VIII because we had not then learned how to deal with fractions and fractional equations.

This chapter will deal with quantities which change together. In addition to what you already know about these **varying** quantities, we shall now study just **how** these quantities vary. For example, does an **increase** in the value of one varying quantity cause a **corresponding increase** in the related quantity? Or does an **increase** in one varying quantity cause a **corresponding decrease** in the other? Can these be expressed (1) graphically, or (2) by tables, or (3) by formulas? These are the points which will be studied in the chapter.

Section 149. Variables and constants. In our study of time, rate, and distance problems we saw that the distance traveled by a train running at any given rate **changes** or **varies** as the time which it has been running **changes** or **varies.** If a train runs at the rate of 40 miles per hour, its movement is described by the equation

$$d = 40\,t.$$

In this equation, *d* and *t* change as the train progresses along its journey. The value of *d* **depends upon** the value of *t*. This means that the **distance and time are variables,** while the **rate is constant.**

Table 20 shows the **tabular method** of representing the relation between these related variables. This shows that

TABLE 21

If the no. of hrs. is	1	2	3	4	5	8	10	15	20
then the distance is	40	80	120	160	200	320	400	600	800

a change in the time causes a change in the distance, or that **a change in one variable causes a change in the related variable.**

EXERCISE 131

1. In the above table, does an *increase* in number of hours always cause an *increase* in distance?
2. In the same table, find the ratio of each distance to its corresponding time. How do these ratios compare? Do the ratios change?
3. A man buys a railroad ticket at 3 cents per mile. Show by the tabular method the relation between cost and number of miles traveled. Show from the table that as the distance increases the cost increases, but that the *ratio* of the cost to the distance does not change. What equation will show the same thing the table shows?
4. Write the equation for the cost of any number of pounds of sugar at 9 cents per pound. What are the variables in your equation? Tabulate the cost for 1, 2, 5, 8, and 10 pounds. Show

from the table that the **ratio** of cost to number of pounds does not change; that is, it is *constant.*

5. A rectangle has a fixed base, 5 inches. Its altitude is subject to change. Tabulate its area if its altitude is 4, 6, 8, 10, and 12 inches. Compare the ratio of any two values of the area with the ratio of the two corresponding values of the altitude. If one altitude is three times another altitude, the one area is __?__ times the other area. Write the equation for its area.
6. A bicyclist rides 10 miles per hour. Show, by three methods, the relation between the number of miles he travels and the number of hours required. In 6 hours he travels __?__ times as far as he travels in 8 hours.

I. DIRECT VARIATION, OR DIRECT PROPORTIONALITY: THE STUDY OF VARIABLES WHICH ARE DIRECTLY RELATED

Section 150. The problems in the previous exercise illustrate **direct variation, or direct proportionality**. In each of the examples, one of the variables depended upon another variable for its value, and the **ratio of any two values of one variable was equal to the ratio of the two corresponding values of the other variables.** When two variables are related in this way, **one is said to vary as, or to be directly proportional to, the other.** Thus, to prove that two variables are **directly proportional, or vary directly,** we must show that

The ratio of any two particular values of one variable is equal to the ratio of the two corresponding values of the other variable.

EXERCISE 132

1. **Illustrative example.** A man earns $6 per day. Show that the amount he earns is *directly proportional* to the number of days he works.

Solution:

(1) $A = 6d$. (We write the equation first, from the conditions of the problem.)

(2) Tabulating:

TABLE 22

If d is	1	2	5	8	10	12
then A is	6	12	30	48	60	72

(3) Now select any two values of A, say 12 and 60, and the two *corresponding* values of d, which are 2 and 10. If the ratio of these two values of A is equal to the ratio of these two values of d, then in the equation $A = 6d$ we know that A is directly proportional to d, or that A varies directly as d. Does $\frac{12}{60} = \frac{2}{10}$? Yes.
Thus, A *is directly proportional* to d, or the amount a man earns at $6 per day is directly proportional to the number of days he works. This is often written $\frac{A_1}{A_2} = \frac{d_1}{d_2}$. A_1 means some particular value of A, and A_2 means some other particular value of A; d_1 and d_2 mean those particular values of d which *correspond* to the selected values of A_1 and A_2.

2. Write the equation for the area of a rectangle whose base is 10 inches. Then show by selecting particular values of A and h that the area is *directly proportional* to the altitude. In other words show that

$$\frac{A_1}{A_2} = \frac{h_1}{h_2}.$$

3. Write the equation for the circumference, C, of a circle whose diameter is D. Is C directly proportional to D? Why?

4. Show that the perimeter of a square is directly proportional to the length of its side.

5. Show that the interest on \$1000 at 6% is directly proportional to the time.

6. x varies directly as y, and when $x = 10, y = 2$. Find the value of x when $y = 7$.

7. C varies directly as d, and when $d = 12$, $c = 38$. What is d when $c = 72$?

8. Is your grade in mathematics directly proportional to the amount of time you spend in preparing your lessons?

9. Is the cost of a pair of shoes directly proportional to the size?

10. P and s represent the perimeter and side, respectively, of a square. What is the meaning of the statement

$$\frac{P_1}{P_2} = \frac{s_1}{s_2}?$$

Write this as a general truth. See Example 4, above.

11. In Example 7 it was stated that the circumference, C, varies directly as the diameter, d. Would it be just as well to state it

$$\frac{C_1}{C_2} = \frac{d_1}{d_2}?$$

It is very often expressed in this form. Find C_2 if $C_1 = 25$, $d_1 = 8$, and $d_2 = 12$.

12. C represents the cost of a railroad ticket, and d represents the number of miles traveled. What is the meaning of

$$\frac{C_1}{C_2} = \frac{d_1}{d_2}?$$

13. Express algebraically that x varies directly as y.

14. If i represents the interest on a certain sum of money, and r stands for the rate, what is the meaning of the expression

$$\frac{i_1}{i_2} = \frac{r_1}{r_2}?$$

15. In a science class the topic of the evaporation of moisture was being discussed. The *amount* of evaporation that would take place in a given amount of time (the temperature remaining constant) was the particular problem. It was finally stated as follows:

$$\frac{A_1}{A_2} = \frac{t_1}{t_2}.$$

What did they mean by this?

16. In studying the relation between the *weight* of water in a tank and its *volume*, a class used the expression

$$\frac{W_1}{W_2} = \frac{V_1}{V_2}.$$

Interpret this statement. Find W_2 if $W_1 = 625$, $V_1 = 10$, and $V_2 = 15$.

17. The relation between the weight of a piece of wire and its length is expressed by the expression

$$\frac{W_1}{W_2} = \frac{l_1}{l_2}.$$

State this relation in words.

18. Express algebraically that the stretch, S, on a pair of spring scales, varies directly as the weight W. If a weight of 12 pounds produces a stretch of $\frac{1}{2}$ inch, how much weight produces a stretch of $\frac{2}{3}$ inch?

19. A train runs at a constant rate of speed. State algebraically that the distance, d, which it travels, is directly proportional to the time, t, which it has been running.

20. The number of gallons of gasoline consumed by an automobile is directly proportional to the distance traveled. State the same fact symbolically.

21. W varies directly as V, and when $W = 12$, $V = 5$. Find V when $W = 40$.

Section 151. One variable may vary as the square of, or the cube of, another variable. It frequently happens that two variables are so related that a change in one of them is accompanied by, or causes **a greater change in, the other.** That is, if one of the variables is doubled, the other related variable may be made four times as large, or eight times as large. An illustration will make this clear. Think of squares whose sides are of different length. Compare their sides, and then compare their corresponding areas. A tabulation will help.

TABLE 23

If s is	2	3	4	5	6	7	8	10	12	15	20	etc.
then A is	4	9	16	25	36	49	64	100	144	225	400	etc.

Select any two particular values of A, such as 9 and 25, and the corresponding values of s, 3 and 5. Does

the ratio of the two values of A, 9 and 25, equal the ratio of the corresponding values of s, 3 and 5? That is, does

$$\tfrac{9}{25} = \tfrac{3}{5}? \quad \text{No.}$$

That is, does
$$\frac{A_1}{A_2} = \frac{s_1}{s_2}?$$

Evidently not. But $\dfrac{9}{25}$ does equal $\dfrac{3^2}{5^2}$; that is

$$\frac{A_1}{A_2} \text{ does equal } \frac{s_1^{\,2}}{s_2^{\,2}}.$$

This shows that the ratio of any two particular values of A is equal to the ratio of the squares of the corresponding values of s; or, that the area of a square is directly proportional to the square of its side. Note the algebraic statement of this relation,

$$\frac{A_1}{A_2} = \frac{s_1^{\,2}}{s_2^{\,2}}.$$

A study of this illustration leads to the general statement:

One variable quantity is directly proportional to, or varies directly as, the square of another variable quantity, when the ratio of any two particular values of the first variable is equal to the ratio of the squares of the corresponding values of the second variable. Expressed in algebraic form this becomes

$$\frac{A_1}{A_2} = \frac{b_1^{\,2}}{b_2^{\,2}}.$$

EXERCISE 133

1. Express algebraically that the area of a circle varies directly as the square of the radius.

2. Assign at least 10 particular values to s in the formula for the area, A, of a square of side s, and compute the corresponding values of A. Tabulate these data. Then make a graph of this relation. Plot the values of s on the horizontal axis.

3. The distance, d, which an object falls in any time, t, varies directly as the square of the time. (t must be measured in seconds.) Make the algebraic statement for this law.

4. Given that $\frac{d_1}{d_2} = \frac{t_1^2}{t_2^2}$, find d_2 when $d_1 = 400$, $t_1 = 5$, and $t_2 = 9$.

5. If the radius of one circle is twice that of another, then the area of one is __?__ times the other. See Example 1.

6. It is proved in geometry that the areas of two similar figures are directly proportional to the squares of any two corresponding sides. Using A_1 and A_2 as the areas of the similar triangles shown in page 82, and s_1 and s_2 as two corresponding sides, make an algebraic statement which will express the fact just stated.

7. Referring to the general principle stated in example 6, we know that if a side of one triangle is twice its corresponding side in another triangle, then the area of the first triangle will be __?__ times the area of the second triangle. How does A_1 compare with A_2 if $s_1 = 2$ and $s_2 = 6$? If $s_1 = 5$ and $s_2 = 10$? If $s_1 = 4$ and $s_2 = 8$?

8. A baker sells two sizes of pies (the thickness is the same in both); one kind is 4 inches in diameter and the other is 8 inches. If the small ones are worth 10¢ each, what are the large ones worth?

9. The volumes of two similar shaped objects are directly proportional to the cubes of any two corresponding dimensions, *i.e.*

$$\frac{V_1}{V_2} = \frac{d_1^3}{d_2^3}.$$

(*a*) A bucket 8 inches deep will hold __?__ times as much water as a similar one 4 inches deep?

(*b*) A 10-inch cube of granite is how many times as large as a 5-inch cube?

(*c*) Think of two spheres, one 4 inches in diameter, and the other 2 inches. The large one is __?__ times as large as the small one?

(*d*) Find V_2 if $V_1 = 125$, $d_1 = 5$, and $d_2 = 3$.

10. How much per dozen could you afford to pay for oranges 4 inches in diameter, if oranges 2 inches in diameter sell for 25¢ per dozen? (Assume the same quality for both kinds.)

11. If an iron ball 3 inches in diameter weighs 12 pounds, how much will one 5 inches in diameter weigh?

12 Suppose it cost 20¢ to buy enough leather to cover a 3-inch cube. What would enough leather to cover a 6-inch cube cost? Explain.

13. x varies directly as the square of y; when $x = 2$, $y = 3$. Find x when $y = 5$.

II. INVERSE VARIATION: THE STUDY OF VARIABLES WHICH ARE INVERSELY RELATED

Section 152. When quantities are inversely related to each other. In the previous exercise the varying quantities were so related in any particular problem that an increase in one variable caused a corresponding increase in the other variable. Some variables, however, are so related that an **increase** in one is accompanied by a corresponding **decrease** in the other.

An example: An increase in the rate at which a train moves causes a decrease in the time required to travel a certain distance. If the train travels at the rate of 20 miles per hour, it will require 5 hours to cover 100 miles; but if it **increases** its rate to 30 miles per hour, it will **decrease** the time so that only $3\frac{1}{3}$ hours will be required to make the trip.

Let us illustrate this fact more in detail by tabulating the relation between the rate and the time of a train which makes a trip of 100 miles. Note from the table how a change in **one variable**, say the rate, is accompanied by a change in the **other variable**, the time.

TABLE 24

If the rate is	10	$12\frac{1}{2}$	15	20	25	30	$33\frac{1}{3}$	40	50
then the time is	10	8	$6\frac{2}{3}$	5	4	$3\frac{2}{3}$	3	$2\frac{1}{2}$	2

This shows that an **increase** in the rate is accompanied by a **decrease** in the time. If we select **any** two values of the rate, say 20 and 50, and the **corresponding** values of the time, 5 and 2, we see that the ratio of the two values of the

rate $\frac{20}{50}$ is **not** equal to the ratio of the corresponding values of the time $\frac{5}{2}$. Clearly, $\frac{20}{50}$ does **not** equal $\frac{5}{2}$, or, to use the more general form,

$$\frac{r_1}{r_2} \text{ does not equal } \frac{t_1}{t_2}.$$

These ratios **would** be equal, however, if we should invert one of them, *e.g.*

$$\frac{20}{50} = \frac{2}{5} \text{ or } \frac{r_1}{r_2} = \frac{t_2}{t_1}.$$

The fact that the ratio of any two values of one of the variables is equal to the **inverted** ratio of the corresponding values of the other variable leads us to say that one of them is **inversely proportional** to the other, or **varies inversely** as the other.

This gives the following principle:

One variable is inversely proportional to another when the ratio of any two values of one of them is equal to the INVERTED RATIO of the two corresponding values of the other.

EXERCISE 134

1. The area of a rectangle is 200 sq. ft. Give several pairs of numbers that might represent its base and altitude. Then show that the ratio of any two particular values of the base is equal to the inverted ratio of the corresponding values of the altitude. In other language show that

$$\frac{b_1}{b_2} = \frac{a_2}{a_1}.$$

2. Ten men can do a piece of work in 32 days. Would an increase in the number of men cause an increase in the number of days? If m rep-

resents the number of men, and d the number of days, does d vary as m varies? does d increase as m increases? Suppose that 10 men could do the work in 32 days, or 20 men could do it in 16 days. From this fact, could we say that $\frac{d_1}{d_2}=\frac{m_1}{m_2}$? Why not? Or that $\frac{d_1}{d_2}=\frac{m_2}{m_1}$? This study shows that the number of days required to do a piece of work is $\underline{\ ?\ }$ proportional to the number of men employed.

3. Two variables, x and y, are inversely proportional; *i. e.* $\frac{x_1}{x_2}=\frac{y_2}{y_1}$. Find x if $x = 12.4$, $y=8.2$, and $y = 6.2$.

4. The time required to make a certain trip is inversely proportional to the rate; or, $\frac{t_1}{t_2}=\frac{r_2}{r_1}$. If the rate is 20 miles per hour, the time will be 8 hours. What is the time required if the rate is 15 miles per hour?

Tell what kind of variation or proportionality is represented by each of the following variables:

5. The amount of flour which you can buy for a dollar, and the price per pound.

6. The size of a 10 ¢ loaf of bread, and the price of flour.

7. The distance you travel on a railroad, and the carfare paid.

8. The value of a fraction with a constant denominator, and its variable numerator.

9. The value of a fraction with a constant numerator, and its variable denominator.

10. The time it takes a boy to run a 100-yard dash, and his rate.

Section 153. Graphical method of representing inverse variation. Figures 138 and 139 show graphically the relation between two numbers which are inversely proportional, or which vary inversely. It represents the base and altitude of a rectangle whose area is always constant, say 100 sq. ft.

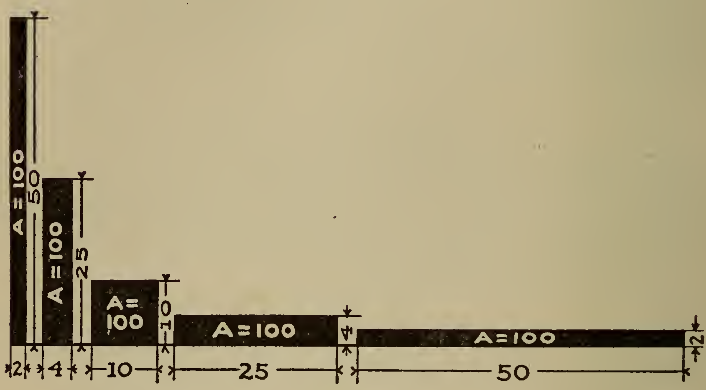


FIG. 138

This graph shows a series of rectangles of constant area, but with variable dimensions. It is very important to note that as the bases increase, the altitudes decrease.

A more frequently used method of the graphic representation of this inverse variation is illustrated in the next figure.

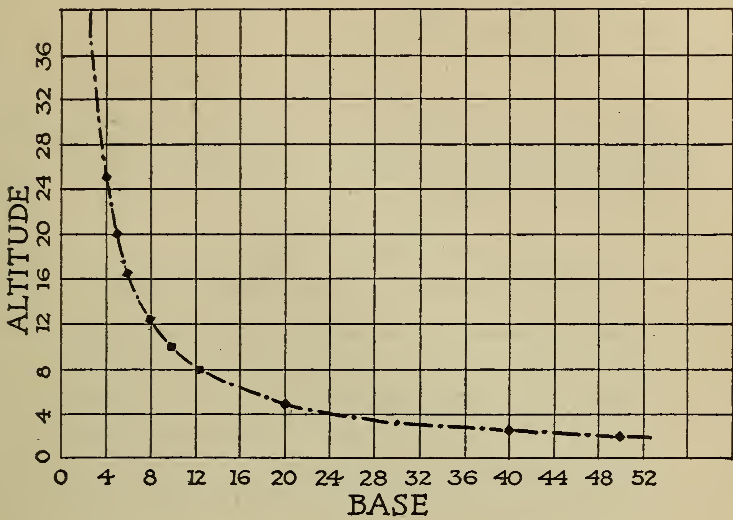


FIG. 139. **The line shows the relationship between two numbers which vary INVERSELY; in this case the relationship between the altitude and base of a rectangle whose area is constant, say 100 sq. ft. As the altitude INCREASES, the base DECREASES.**

To construct this graph, the following table was made:

TABLE 25

If base is	2	4	5	6	8	10	12.5	20
then altitude is	50	25	20	16.6	12.5	10	8	5

Note that as the base *increases*, the altitude *decreases*. How does the graph show this relation? In what way does this graph differ from those you have previously dealt with?

Show that the equation

$$\frac{a_1}{a_2}=\frac{b_2}{b_1}$$

describes the relation between the base and altitude of any rectangle whose area is constant, say 100 sq. ft.

EXERCISE 135

GRAPHICAL REPRESENTATION OF INVERSE VARIATION

1. The product of two variables, x and y, is always 200. Tabulate 10 pairs of values of these variables, and from the table construct a graph showing the way in which the variables are related. Measure values of x along the horizontal axis.
2. Some tourists decide to make a trip of 100 miles. Show graphically the relation between (1) the different rates at which they might travel, and (2) the time required at each rate.
3. Think of some good illustration of inverse variation. Represent it graphically.

REVIEW EXERCISE 136

1. If 60 cu. in. of gold weighs 42 lb., how much will 35 cu. in. weigh?
2. If a section of a steel beam 10 yd. long weighs 840 lb., how long is a piece of the same material which weighs 1250 lb.?
3. At 40 lb. pressure per square inch, a given pipe discharges 160 gal. per minute. How many gallons per minute would be discharged at 65 lb. pressure?

4. A steam shovel can handle 900 cu. yd. of earth in 7 hr. At the same rate how many cubic yards can be handled in 5 hr.?

5. A train traveling at the rate of 50 miles per hour covers a trip in 5 hours. How long would it take to cover the same distance if it traveled at the rate of 35 miles per hour?

6. If 50 men can build a boat in 20 days, how long would it take 30 men to build it?

7. A wheel 28 in. in diameter makes 42 revolutions in going a given distance. How many revolutions would a 48-inch wheel make in going the same distance?

8. The volume, v, of a gas is inversely proportional to its pressure, p. Write an equation showing this fact.

9. If the volume of a gas is 600 cubic centimeters (cc.) when the pressure is 60 grams per square centimeter, find the pressure when the volume is 150 cubic centimeters.

10. When are two changing quantities or variables directly proportional? When do they vary inversely?

11. How can you test for direct variation? for inverse variation? Are x and y *directly* proportional in the equation $x = 2y + 5$?

12. In order to save d dollars in n years, how much would your savings have to average per month?

13. The edge of one cube is x, and of another $2x$. What is the ratio of their surfaces? Of their volumes?

14. The radii of two spheres are r and $3r$. What is the ratio of their volumes?

15. The ratio of the diameter of the earth and the sun is $\frac{1}{108}$. The volume of the sun is how many times as large as the earth?

16. The weight and distance (in a teeter board) are inversely proportional, *i.e.*

$$\frac{W_1}{W_2} = \frac{d_2}{d_1}.$$

Find W_2 if $W_1 = 80$, $d_1 = 10$, and $d_2 = 12$.

17. If it costs $75 to paint the exterior surface of a certain house, how much should it cost to paint a house whose dimensions are twice as great? What principle of mathematics is involved here?

CHAPTER XVII

SQUARE ROOTS AND RADICALS

Section 154. Meaning of square root. In Chapter VI we found the square root of numbers which occurred in connection with the sides of a right triangle. At that time we did not make a special study of square root, but merely referred to the arithmetical method. This chapter will give **a clearer notion of square root.**

The **square root of a number** is defined as **one of its two equal factors.** To illustrate, we might factor 100 in the following ways:

$$4 \times 25 = 100$$
$$5 \times 20 = 100$$
$$10 \times 10 = 100$$
$$8 \times 12.5 = 100, \text{ etc.}$$

Thus, by finding two equal factors, we see that the square root of 100 is 10. We shall use this definition constantly: **the square root of a number is one of its two equal factors.**

In the same way the cube root of a number is one of the three equal factors of the number. Thus, -2 is the cube root of -8 because $(-2)(-2)(-2) = -8$.

Section 155. A number has two square roots. Just as 10 was a square root of 100 because $10 \times 10 = 100$, so -10 is also a square root of 100, because $(-10)(-10) = 100$. In the same way, either $+a$ or $-a$ is the square root of a^2, because $(a)(a) = a^2$ and $(-a)(-a) = a^2$.

Section 156. How to indicate the root of a number. It has been agreed to indicate the root of a number by the symbol $\sqrt{}$, called a **radical sign.** To designate **what** root is meant, a small number called an **index** is placed in the radical sign. Thus, $\sqrt[4]{16}$ means the fourth root of 16, *i.e.* 2, **because** $2 \cdot 2 \cdot 2 \cdot 2 = 16$; $\sqrt[3]{27}$ means the cube

root of 27, *i.e.* 3, because $3 \cdot 3 \cdot 3 = 27$; and $\sqrt[2]{25}$ means the square root of 25. It is customary, however, to omit the index when **square** root is meant. Thus, $\sqrt{25}$, without an index, is always understood to mean the **square** root of 25.

EXERCISE 137

FIND, BY TRIAL, THE ROOTS, WHICH ARE INDICATED, OF THE FOLLOWING EXPRESSIONS

1. $\sqrt{9}$

2. $\sqrt{x^6}$

3. $\sqrt{25\,a^2}$

4. $\sqrt[3]{8}$

5. $\sqrt[3]{27\,x^3}$

6. $\sqrt[3]{b^{12}}$

7. $\sqrt{64\,x^8}$

8. $\sqrt{100\,y^6}$

9. $\sqrt{\frac{1}{4}}$

10. $\sqrt{\frac{9}{64}}$

11. $\sqrt{\dfrac{36\,x^2}{49\,y^2}}$

12. $\sqrt[3]{\frac{8}{27}}$

13. $\sqrt{144\,a^{10}}$

14. $\sqrt{400\,x^2y^4}$

15. $\sqrt[3]{125\,b^6c^9}$

16. $\sqrt[4]{16\,x^8}$

17. $\sqrt[5]{243\,y^{10}}$

18. $\sqrt[4]{256\,x^{12}}$

Section 157. How to find the square root of algebraic expressions. We have seen that $(a+b)^2$ or $(a+b)(a+b) = a^2 + 2\,ab + b^2$. From this *it is evident* that the square root of $a^2 + 2\,ab + b^2$ must be $a + b$, or $\sqrt{a^2 + 2\,ab + b^2} = (a+b)$. In the same way $\sqrt{x^2 + 10\,x + 25}$ is $x + 5$, because $(x+5)(x+5)$ gives $x^2 + 10\,x + 25$. From these illustrations we see that it is possible to extract the square root of an algebraic expression if we can show that it can be obtained by **squaring** some other expression; that is, if we can show that it is the product of **two equal factors**.

EXERCISE 138

Find the square root of the following expressions, where it is possible to do so. Check each.

1. $x^2 + 2xy + y^2$
2. $a^2 + 6a + 9$
3. $b^2 - 4b + 4$
4. $t^2 - 10t + 25$
5. $4a^2 + 12a + 9$
6. $16 + x^2 + 8x$
7. $1 + 21x + 100x^2$
8. $y^2 + 6y + 20$
9. $25a^2 + 40a + 16$
10. $x^2 + 16$
11. $y^2 - 49$
12. $12x + 36 + x^2$
13. $t^2 + u^2 + 2tu$
14. $r^2s^4 - 6rs^2 + 9$

If any of the expressions above are not perfect squares, make the necessary changes to transform them into perfect squares.

This chapter teaches how to find the square root of only one kind of algebraic expression, namely, an expression which can be shown to be the product of two equal factors, or the square of an expression.

Thus, to find the square root of an algebraic expression, you must show that it has been made by squaring another expression. If it is the product of two unequal factors, then we cannot, in this course, find the square root.

EXERCISE 139

FURTHER PRACTICE IN FINDING THE SQUARE ROOT OF ALGEBRAIC EXPRESSIONS

1. $x^2 - 8x + 16$
2. $p^2 + 12p + 36$
3. $9x^2 + 30x + 25$
4. $4y^2 - 20y + 25$
5. $p^2 + p + \frac{1}{4}$
6. $c^2 + \frac{2}{3}c + \frac{1}{9}$
7. $25b^2 + 5b + \frac{1}{4}$
8. $16y^2 - 10y + 1$
9. $8x + x^2 + 15$
10. $r^4 - 6r^2s^2 + 9s^4$

11. $x^2 - \frac{5}{2}x + \frac{25}{16}$
12. $b^2 + \frac{3}{4}b + \frac{9}{16}$
13. $y^2 + \frac{1}{2}y + \frac{1}{4}$
14. $x^2 + x + 4$
15. $y^2 - \frac{2}{3}y + \frac{1}{9}$
16. $4a^2 + 25$
17. $1 + 9x^2 + 6x$
18. $\frac{1}{4} + b^2 + b^4$

Section 158. Equations solved by finding square roots. We have seen that $(+3)(+3) = 9$, and also $(-3)(-3) = 9$, and hence that 9 has two square roots, $+3$ and -3. In solving the equation

$$x^2 = 9$$

we are to find a number, or numbers, the square of which is 9. Evidently we find the square root of each side of the equation, giving $x = +3$ or -3. (This is commonly written ± 3.)

EXERCISE 140

Find two values of the unknown which will satisfy each of the following equations.

1. $x^2 = 16$
2. $x^2 = 64$
3. $x^2 - 100 = 0$
4. $x^2 = 4a^2$
5. $y^2 = 81b^4c^2$
6. $x^2 = 2^2a^2$
7. $x^2 = a^2 + 6a + 9$
8. $y^2 = 4b^2 + 4b + 1$
9. $x^2 - 49 = 0$
10. $y^2 - 25a^6b^4 = 0$
11. $x^2 = (a+b)^2$
12. $y^2 = c^2 + 2cd + d^2$
13. $x^2 = 100(a-b)^2$
14. $x^2 = \frac{4}{25}$
15. $x^2 - b^2 = 0$
16. $x^2 = 1 \cdot 49$

Section 159. Square root of fractions. From the fact that $(\frac{2}{3})^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$, we see that the square root of $\frac{4}{9}$ must be $\frac{2}{3}$. In the same way, the square root of $\frac{a^2}{b^2}$ must be $\frac{a}{b}$, because $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$. This suggests the principle that

The square root of a fraction equals the square root of the numerator divided by the square root of the denominator.

If the numerator or denominator of the fraction is not a perfect square, the approximate square root of the fraction is found in the same way, but the process is very long and unnecessarily laborious. To illustrate, the square root of $\frac{2}{5}$ by this method gives

$$\sqrt{\frac{2}{5}}=\frac{\sqrt{2}}{\sqrt{5}}=\frac{1.414}{2.236}=.632.$$

EXERCISE 141

Find the square root of $\frac{3}{7}$; of $\frac{5}{8}$; of $\frac{2}{3}$.

Section 160. Easier methods of finding the square root of a fraction. Consider the same example given in Section **159**: $\sqrt{\frac{2}{5}}$. Either of two methods will be much shorter than the previous method.

1. Reducing the common fraction to a decimal fraction.

$$\tfrac{2}{5}=\sqrt{.4}=.632.$$

This calls for the square root of only one number, a decimal.

2. Making denominator a perfect square.

$$\sqrt{\frac{2}{5}}=\sqrt{\frac{10}{25}}=\frac{\sqrt{10}}{5}=\frac{3.162}{5}=.632.$$

The advantage of the second method is that it makes you find the square root of only one number, which is an integer rather than a decimal.

This method is based upon **the principle** that both numerator and denominator may be multiplied by the same number without changing the value of the fraction. You

must always multiply by a number that will make the denominator a perfect square. Why?

Which is the simpler form for computation?

(*a*) $\sqrt{\frac{2}{3}}$ or $\sqrt{\frac{6}{9}}$ or $\frac{\sqrt{6}}{3}$? (*b*) $\sqrt{\frac{3}{5}}$ or $\sqrt{\frac{15}{25}}$ or $\frac{\sqrt{15}}{5}$?

(*c*) $\sqrt{\frac{3}{7}}$ or $\sqrt{\frac{21}{49}}$ or $\frac{\sqrt{21}}{7}$? (*d*) $\sqrt{\frac{1}{2}}$ or $\sqrt{\frac{2}{4}}$ or $\frac{\sqrt{2}}{2}$?

(*e*) $\sqrt{\frac{1}{3}}$ or $\sqrt{\frac{3}{9}}$ or $\frac{\sqrt{3}}{3}$?

It should now be clear that the best method of finding the square root of a fraction is that of:

1. **Multiplying numerator and denominator by a number which will make the denominator a perfect square;**
2. **Then finding the approximate square root of the numerator and dividing the result by the square root of the denominator.**

EXERCISE 142

PRACTICE IN FINDING THE SQUARE ROOT OF FRACTIONS

Compute to three decimal places.

1. $\sqrt{\frac{2}{3}}$	**6.** $\sqrt{\frac{1}{4}}$	**11.** $\sqrt{\frac{5}{3}}$	**16.** $\sqrt{\frac{3}{12}}$
2. $\sqrt{\frac{3}{5}}$	**7.** $\sqrt{\frac{1}{5}}$	**12.** $\sqrt{\frac{7}{2}}$	**17.** $\sqrt{\frac{2}{8}}$
3. $\sqrt{\frac{1}{2}}$	**8.** $\sqrt{\frac{2}{7}}$	**13.** $\sqrt{\frac{4}{7}}$	**18.** $\sqrt{\frac{8}{5}}$
4. $\sqrt{\frac{1}{3}}$	**9.** $\sqrt{\frac{3}{7}}$	**14.** $\sqrt{\frac{7}{3}}$	**19.** $\sqrt{\frac{2}{11}}$
5. $\sqrt{\frac{2}{5}}$	**10.** $\sqrt{\frac{2}{9}}$	**15.** $\sqrt{\frac{11}{5}}$	**20.** $\sqrt{\frac{1}{10}}$

21. Find the square root, to three decimal places, of 2, 3, and 5. Fix these definitely in your memory. They will serve you well later.

Section 161. The square root of a product, one factor of which is a perfect square. The labor of finding square roots is greatly reduced by the use of a principle which is illustrated here.

(*a*) $\sqrt{12}=\sqrt{4\cdot 3}=\sqrt{4}\cdot\sqrt{3}=2(1.732)=3.464.$

(*b*) $\sqrt{75}=\sqrt{25\cdot 3}=\sqrt{25}\cdot\sqrt{3}=5(1.732)=8.660.$

(*c*) $\sqrt{a^2b}=\sqrt{a^2}\cdot\sqrt{b}=a\sqrt{b}.$

Note here that the number whose square root is desired is the product of two factors, one of which is a perfect square. The principle used here is usually stated:

The square root of a product is equal to the product of the square roots of the factors, or

$$\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}.$$

In applying this principle, note that the number whose square root is desired must be separated into two factors, one of which is a perfect square. The square root of this factor must then be multiplied by the approximate square of the other factor. For example:

(*a*) $\sqrt{200}=\sqrt{100\cdot 2}=\sqrt{100}\cdot\sqrt{2}=10(1.414)=1.414.$

(*b*) $\sqrt{20}=\sqrt{4\cdot 5}=\sqrt{4}\cdot\sqrt{5}=2\sqrt{5}=2(2.226)=4.452.$

Those students who know the square root of 2, 3, and 5 will have a great advantage in this work.

EXERCISE 143

Solve by the short method.

1. $\sqrt{12}$	**4.** $\sqrt{108}$	**7.** $\sqrt{32}$	**10.** $\sqrt{72}$
2. $\sqrt{75}$	**5.** $\sqrt{27}$	**8.** $\sqrt{18}$	**11.** $\sqrt{20}$
3. $\sqrt{300}$	**6.** $\sqrt{50}$	**9.** $\sqrt{8}$	**12.** $\sqrt{45}$

13. $\sqrt{80}$ **15.** $\sqrt{125}$ **17.** $\sqrt{288}$ **19.** $\sqrt{150}$

14. $\sqrt{500}$ **16.** $\sqrt{98}$ **18.** $\sqrt{147}$ **20.** $\sqrt{54}$

21. Use this method of square root in finding the diagonal of a square when each side is 10; 5; 8; 20; 12.

RADICALS

Section 162. Definition of radical. An indicated root of any expression is called a **radical.** The indicated square root of 10, *i.e.* $\sqrt{10}$, is a **radical.** Similarly $\sqrt{150}$, $\sqrt{x}$, $\sqrt{x^2 + 10x + 25}$, $\sqrt[3]{8}$, etc., are **radicals.**

Section 163. Radicals may be expressed in different forms. The form of a radical expression may be changed without changing its numerical value. Recall that in Section 161 we changed the form of radicals to put them in a more convenient form for the computation of square roots; for example, $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$. Here we have three different forms, each having the same numerical value. Also, recall that we changed $\sqrt{\frac{2}{5}}$ to $\sqrt{\frac{10}{25}}$ and then to $\frac{\sqrt{10}}{5}$, giving us three different forms of the same radical.

Section 164. Radicals should be reduced to the simplest form for computation. By comparing the following radical forms you will be able to state when a radical is in its simplest or most convenient form for computation.

Which is most easily computed:

(*a*) $\sqrt{50}$ or $\sqrt{10 \cdot 5}$ or $\sqrt{25 \cdot 2}$ or $5\sqrt{2}$?

(*b*) $\sqrt{32}$ or $\sqrt{8 \cdot 4}$ or $\sqrt{16 \cdot 2}$ or $4\sqrt{2}$?

(*c*) $\sqrt{\frac{1}{2}}$ or $\frac{\sqrt{1}}{\sqrt{2}}$ or $\sqrt{\frac{2}{4}}$ or $\frac{\sqrt{2}}{2}$?

(*d*) $\sqrt{\frac{3}{5}}$ or $\frac{\sqrt{3}}{\sqrt{5}}$ or $\sqrt{\frac{15}{25}}$ or $\frac{\sqrt{15}}{5}$?

These examples suggest the following important **principle** concerning the simplest form of radicals:

A radical is in its simplest form when it contains no factor of the same power as the root indicated, and when no radical occurs in a denominator.

For example, $\sqrt{\frac{a}{b}}$ is not in its simplest form, for the radical $\sqrt{b}$ occurs in the denominator. This is reduced to the simplest form by multiplying both numerator and denominator by an expression which will make the denominator a perfect square. Thus

$$\sqrt{\frac{a}{b}}=\sqrt{\frac{ab}{b^2}}=\frac{\sqrt{ab}}{b}.$$

The last radical is in its simplest form. Also, the radical $\sqrt{a^2b}$ is not in its simplest form, because the factor a^2 is of the same power as the root. To reduce it to the simplest form we must take the square root of the a^2, which gives $\sqrt{a^2b}=a\sqrt{b}$. The last radical is in its simplest form.

Which of the following radicals are in their simplest form:

$\sqrt{\frac{1}{2}}$, $\frac{\sqrt{2}}{3}$, $\frac{\sqrt{a}}{b}$, $\sqrt{\frac{a}{b}}$, $\frac{\sqrt{3}}{\sqrt{5}}$, $\frac{\sqrt{5}}{2}$, $\frac{1}{3}\sqrt{5}$, $\sqrt{8}$, $\sqrt{50}$, $\sqrt{7}$, $\sqrt{10}$, $\sqrt{12}$, $\sqrt{20}$, $\sqrt{a^4b}$, $\sqrt{ab^2}$, $\sqrt{ab}$, $2\sqrt{8}$, $a\sqrt{a^2c}$, and $b\sqrt{ab}$.

EXERCISE 144

PRACTICE IN CHANGING RADICALS TO THEIR SIMPLEST FORM

Do not find numerical values.

1. $\sqrt{8}$	**3.** $\sqrt{12}$	**5.** $\sqrt{200}$	**7.** $\sqrt{xy}$
2. $\sqrt{10}$	**4.** $\sqrt{20}$	**6.** $\sqrt{ab}$	**8.** $\sqrt{25\,xy}$

9. $\sqrt{8ab}$

10. $\sqrt{50xy}$

11. $\sqrt{\frac{1}{2}}$

12. $\sqrt{\frac{2}{3}}$

13. $\sqrt{\frac{3}{5}}$

14. $\sqrt{\frac{1}{10}}$

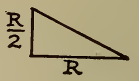
15. $\sqrt{\frac{a}{b}}$

16. $\sqrt{\frac{a}{5}}$

17. $\sqrt{\frac{x}{y}}$

18. $\sqrt{\frac{a}{b}}$

19. The base and altitude of the right triangle shown here are R and $\frac{R}{2}$. Express in simplest radical form the hypotenuse.



20. Find the simplest radical expression for the diagonal of a rectangle whose dimensions are n and $2n$; n and $3n$; $3n$ and $4n$.

21. What is the diagonal of a rectangle whose dimensions are $\frac{1}{2}s$ and s? $\frac{1}{2}s$ and $2s$? $\frac{1}{4}s$ and s?

TIMED PRACTICE EXERCISE J

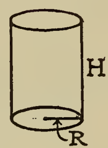
How many examples in this exercise can you do in 3 minutes? The very rapid and accurate pupils in 100 cities did 16; the average about 10.

1. $\sqrt{8}$	8. $\sqrt{x^5y^8}$	15. $\sqrt{\frac{2}{7}}$
2. $\sqrt{a^3b^4}$	9. $\sqrt{\frac{2}{5}}$	16. $\sqrt{18}$
3. $\sqrt{\frac{2}{3}}$	10. $\sqrt{50}$	17. $\sqrt{r^7s^4}$
4. $\sqrt{27}$	11. $\sqrt{a^5b^4}$	18. $\sqrt{\frac{3}{11}}$
5. $\sqrt{x^3y^6}$	12. $\sqrt{\frac{3}{2}}$	19. $\sqrt{24}$
6. $\sqrt{\frac{3}{10}}$	13. $\sqrt{45}$	20. $\sqrt{b^7c^{10}}$
7. $\sqrt{12}$	14. $\sqrt{n^5p^6}$	21. $\sqrt{\frac{5}{2}}$

REVIEW EXERCISE 145

1. The area of a circle is expressed by the formula, $A = \pi R^2$. ($\pi = 3.1416$.) Solve for R. Find R when $A = 3.1416$; when $A = 100$; when $A = \pi$; $A = 500$.

2. The entire area of the cylinder shown here is $2\,\pi R^2 + 2\,\pi RH$, or $2\,\pi R(R + \pi)$. Find the entire area when $R = 3.5$.



3. The area of a circle whose radius is 10 inches is 314.16 sq. in. What is the area of a circle whose radius is twice as great?

4. The sides of a triangle are 12, 28, and 32. The perimeter of a similar triangle is $\frac{3}{4}$ that of the given triangle. Find the sides of the second triangle.

5. The legs, altitude, and base of a right triangle are equal. Find each if the hypotenuse is 30.

6. The legs of a right triangle are $\frac{R}{2}$ and $\frac{3R}{2}$. Find the hypotenuse.

7. The area of a right triangle is 50. If the base and altitude are equal, what is the length of each?

8. One leg of a right triangle is $\frac{R}{2}$ and the hypotenuse is R. Find the other leg.

9. What is the tangent of one of the acute angles of a right triangle, if the legs are equal? Make a drawing.

10. What is the cosine of one of the acute angles of a right triangle whose base and altitude are equal?

11. Draw an equilateral triangle of side s. Solve for the altitude and for the area.

12. Find the altitude of an equilateral triangle whose sides are 10 inches. What is its area?

13. One angle of a right triangle is 30°. The hypotenuse is 20. Find the other two sides.

14. One angle of a right triangle is 30°. The hypotenuse is 20. Find the other two sides. Compare this example with the previous one.

15. The side of a square room is 21.5 feet. Find its diagonal correct to two decimals.

16. What is the perimeter of a square whose diagonal is 12 inches?

17. The side of a square is a. What represents its area? its perimeter? its diagonal?

18. A rectangle is four times as long as it is wide. Find its diagonal if its area is 576 square inches.

19. Figure 141 is an equilateral triangle, and CD is perpendicular to AB. Find CD if each side of the triangle is 20 inches. Then find the area of triangle ABC.

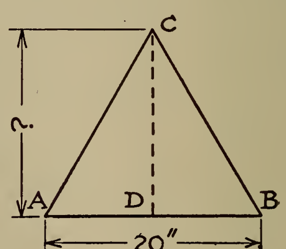


FIG. 140

20. The area of a right triangle is 24 square inches. Its base is 6 inches. Find its altitude and hypotenuse.

21. The diagonal of a square is d. Show that s (side) is $\frac{d}{\sqrt{2}}$.

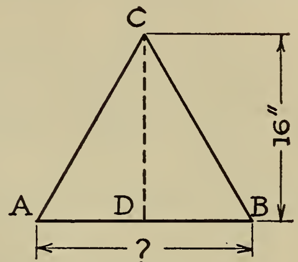


FIG. 141

22. CD, the altitude of equilateral triangle ABC, is 16 inches. Find the sides of the triangle and its area.

23. How long an umbrella will lie flat down on the bottom of a trunk whose inside dimensions are 27 inches by 39 inches?

24. Can a circular wheel 8 feet in diameter be taken into a shop if the shop door is $4\frac{1}{2}$ feet wide and $6\frac{1}{2}$ feet high?

25. If A represents the area of a square, what will represent its perimeter? its diagonal?

26. The sides of a triangle are 12, 16, and 24 inches. Is it a right triangle? Why?

27. If you know two sides of a triangle, can you *always* find its area? Explain.

28. Find the area of a square whose diagonal is 12 inches longer than one of its sides.

29. Will an umbrella 30 inches long lie flat down in a suit case whose inside dimensions are 18 by 25 inches?

30. A rectangle is 12 by 18 inches. How much must be added to its length to increase its diagonal 4 inches?

31. One side of a right triangle is 3 times as long as the other. The hypotenuse contains 30 inches. Find the area of the triangle.

32. Find the diagonal of a square whose side is 20.

33. Make a formula for finding the diagonal of any square.

34. Find the sides of a square whose diagonal is 20.

35. Make a formula for finding the side of any square whose diagonal is known.

36. When is a radical in its simplest form?

37. Factor $12x^2 - 2x - 24$.

38. $\begin{cases} 2y = 4x + 6 \\ 7y + 3x = 4 \end{cases}$

39. It costs the same to sod a square piece of ground at 15 ¢ per square yard as to put a fence around it at 60 ¢ per yard. Find the side of the square.

40. In how many years will $300 double itself at 6 % simple interest?

CHAPTER XVIII

HOW TO SOLVE EQUATIONS OF THE SECOND DEGREE

Section 165. What are quadratic equations? In all the previous chapters you have solved equations of the **first degree**; that is, equations in which the **unknown** (or **unknowns) did not have exponents greater than 1.** This chapter will show how to solve equations of the **second degree**, equations in which the unknown occurs to the **second power.** To illustrate, you will learn how to solve equations such as

$$x^2 + 8x = 20.$$

The fact that the unknown, x, in this equation occurs in the second power or second degree (as, x^2) leads us to speak of the equation as a **second-degree, or quadratic, equation.**

Three ways of solving second-degree or quadratic equations will be explained. These, in the order in which we shall discuss them, are:

I. **Solution by factoring.**
II. **Solution by completing the square.**
III. **Solution by graphical representation.**

Section 166. Our real problem in solving a quadratic equation is to find a value of x which will make the left side equal to the right side. To solve the quadratic equation

$$x^2 - 8x + 12 = 0$$

we are trying to find a value which will make the left side, $x^2 - 8x + 12$, equal to the right side, 0. It helps to think of a quadratic equation (just as we thought of a simple equation in Chapter I) as asking a question: what value, or values, can x have, to make the expression $x^2 - 8x + 12$ equal to **zero**?

To solve the quadratic equation $x^2 - 8x + 12 = 0$, we have to make a **special study of the expression $x^2 - 8x + 12$,**

to find some value for x which will make this expression equal to **zero**. It will help us to recognize that the **value of this expression depends upon what value we give to x.** If x is 1, the value of the expression is 5; if x is 2, the value of the expression is **0**. Thus we have here **two variable quantities: x itself is one variable,** and **the value of the expression $x^2 - 8x + 12$ is the other variable.**

Section 167. The trial method: laborious and not sure. One way to find the desired value of x is to **assign various values to x in the expression** $x^2 - 8x + 12$, hoping to find a value for x which will make $x^2 - 8x + 12 = 0$. In doing this, one class made the tabulation given in Table 26.

TABLE 26

If x is	0	1	2	3	4	5	6	7	-1	-2
then x^2-8x+12 is	12	5	0	-3	-4	-3	0	5	21	32

From this table, do you see any value of x which makes the expression $x^2 - 8x + 12$ equal to zero? If so, that value of x satisfies the quadratic equation $x^2 - 8x + 12 = 0$. How many values of x will make the expression zero? Check them to be certain.

This method is very laborious, and might often fail to give the value of the unknown. It just happens to give two values here, but it might more easily not happen to give them. It is merely a trial method.

I. HOW TO SOLVE QUADRATIC EQUATIONS BY FACTORING

Section 168. An economical method: factoring, to get a product equal to zero. The use of a principle which we already know gives an easy method of solving quadratic

equations. The principle is: *a product of any number of factors is zero, if one of the factors is zero.* Recall that $4 \cdot 5 \cdot 0 = 0$, or that $a \cdot b \cdot c = 0$ if either a, b, or c is 0. Why? Because any number multiplied by 0 is 0. Under what conditions is the product xyz **zero**? Evidently if either x, y, or z is zero; that is, if one of the factors is **0**.

In the same way, the expression $(x-6)(x-2)$ which is the product of two factors, could be 0 if $x-6$ were 0, or $x-2$ were 0.

Illustrative example. Now let us apply this principle to the solution of the quadratic equation $x^2 - 8x + 12 = 0$.

Solution: $x^2 - 8x + 12 = 0.$

In order to solve the equation, we want to get a product equal to zero. Hence we should factor the expression $x^2 - 8x + 12$.

Forming a product (*i.e.* factoring) gives

$$(x-2)(x-6) = 0.$$

Now we have a product equal to zero. But in order to have a product equal to zero, one of its factors must be zero.

Therefore, if $x - 2 = 0$, x must equal 2.

Similarly, if $x - 6 = 0$, x must equal 6.

This method gives as the values of x, 2 and 6, much more easily than by assigning values to x as in Table 26. A second illustrative example will enable you to use this method.

Illustrative example.

Solve the quadratic equation,

$$x^2 + 2x = 48.$$

Solution: $x^2 + 2x = 48.$

We want a product equal to 0; therefore the equation should be in the form of an expression equal to 0,

or $x^2 + 2x - 48 = 0.$

(1) Forming a product,

$$(x+8)(x-6)=0.$$

(2) Making the first factor = 0,

$$x+8=0,$$

or $$x=-8.$$

(3) Making the second factor 0,

$$x-6=0,$$

or $$x=6.$$

Check: $$64-16=48.$$

$$36+12=48.$$

EXERCISE 146

PRACTICE IN SOLVING QUADRATIC EQUATIONS BY FACTORING

1. What values of x will make x^2-5x+6 equal to 0? or in other words, solve the equation $x^2-5x+6=0$.
2. For what values of y does $y^2-7y+14=0$?
3. What values of x will satisfy the equation $x^2+x-20=0$?

Solve each of the following equations:

4. $a^2+9a-22=0$
5. $x^2+7x=18$
6. $y^2-y=20$
7. $b^2-36=0$
8. $c^2-8c=-16$
9. $m^2=m+2$
10. $t^2+t=56$
11. $2y^2+5y-12=0$
12. $\frac{1}{2}x^2+x=12$
13. $\frac{b^2}{2}+\frac{b}{4}=9$
14. $c^2+2c=0$
15. $12=x^2+x$
16. $(x+6)(x-2)=0$
17. $3b^2+7b=-2$
18. $2x^2+5x-3=0$
19. One number is 2 larger than another. Their product is 80. Find each number.
20. The sum of two numbers is 10; the sum of their squares is 52. Find each number.

21. The altitude of a triangle is 4 inches longer than its base. The area is 96 square inches. Find the base and altitude.

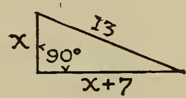
22. The square of a number is 30 more than the number itself. Find the number.

23. The perimeter of a rectangle is 60 inches, and its area is 200 square inches. What are its dimensions?

24. A boy covered a cubical box with paper. He needed 96 sq. in. What is the length of one edge of the cube?

25. The sum of the squares of three consecutive numbers is 50. What are the numbers?

26. Find the length of each side of the triangle shown here.



27. A rectangular floor is 6 ft. longer than it is wide; its area is 34 sq. yd. What is its length?

28. Determine the area of the triangle in example 26.

29. $4x^2 = 81$

30. $x^2 - 9a^2 = 0$

31. $y^2 - 6by + 9b^2 = 0$

32. $x^2 = 8ax - 16a^2$

II. HOW TO SOLVE QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

Section 169. The left side of the equation must be a perfect square. By this method of solving quadratic equations **we find the square root of each side of the equation.** Thus to solve the equation

$$x^2 + 6x = 27$$

we find the square root of each side. But $x^2 + 6x$ is not a perfect square; we cannot extract the square of an algebraic expression which is not a perfect square (the approximate square root of an arithmetical number may be found, but not of an algebraic expression). Therefore, **to use this method**, we must make the left side of the equation a perfect square, or "**complete the square.**" An exercise will recall how to do this. At this point, the examples on pages 268 and 269 should be reviewed.

EXERCISE 147. ORAL WORK

PRACTICE IN MAKING PERFECT SQUARES, BY ADDING CERTAIN TERMS TO GIVEN EXPRESSIONS

1. What is the square of $x + 3$?

2. What is the square of $x + 5$?

3. What does $x^2 + 6x$ lack being a perfect square?

4. What does $y^2 + 10y$ lack being a perfect square?

What should be added in order to make each of the following expressions perfect squares?

5. $x^2 + 4x + ?$

6. $x^2 - 6x + ?$

7. $x^2 + 18x + ?$

8. $y^2 + 5y + ?$

9. $p^2 + 3p + ?$

10. $x^2 + x + ?$

11. $x^2 - 7x + ?$

12. $x^2 - x + ?$

13. $x^2 + \frac{3}{2}x + ?$

14. $x^2 + \frac{1}{2}x + ?$

15. $x^2 - \frac{3}{4}x + ?$

16. $x^2 - \frac{2}{3}x + ?$

The solution of these examples will suggest the following method for completing the square of an expression of the type $\boldsymbol{x^2 + px}$: *i.e.*

Add to the expression the square of $\frac{1}{2}$ of the coefficient of x, *i.e.* $\left(\frac{p}{2}\right)^2$.

Illustrative example, continued. We are now able to solve the equation

$$x^2 + 6x = 27.$$

(1) Adding 9 to each side, to "complete the square,"

$$x^2 + 6x + 9 = 36. \quad \text{(Why 9?)}$$

(2) Extracting the square root of each side gives

$$x + 3 = +6 \text{ or } -6.$$

Using $+6$, $\quad x = 3.$

Using -6, $\quad x = -9.$

Check: $\quad 9 + 18 = 27.$

$\quad 81 - 54 = 27.$

EXERCISE 148

Solve these quadratic equations by "completing the squares."

1. $x^2 + 6x = 40$
2. $x^2 - 8x = 84$
3. $y^2 + 2y = 15$
4. $p^2 - 10p = -16$
5. $t^2 + 3t = 10$
6. $y^2 + 7y = 8$
7. $x^2 + x = 56$
8. $c^2 + 6c = 11$
9. $p^2 - 4p = 36$
10. $10x + x^2 = -9$
11. For what value of x does $x^2 + 6x = 36$? Could this equation be solved by the factoring method?
12. Does $8x$ ever equal $9 - x^2$?
13. A rectangular field is 2 rods longer than it is wide, and it contains 6 acres. Find its width.
14. Find two consecutive even numbers whose product is 48.
15. $x^2 - 4x - 32 = 0$
16. $x^2 = 18$
17. $x^2 - 7x - 18 = 0$
18. $y + 6 = y^2$

19. $p^2 - 5p = 11$
20. $x^2 - 2x = \frac{5}{4}$
21. $x^2 + x = 20$
22. $y^2 + y = 30$
23. $y^2 - 3y = 10$
24. $x^2 + x = \frac{3}{4}$
25. Make a good rule for solving these equations.

Section 170. In order to complete the square the equation must be in such form that the coefficient of x^2 is 1. If the equation is given in such form as $2x^2 + 5x = 12$, it must be changed so that the first term is x^2.

Illustrative example.

Solve $\quad 2x^2 - 5x = 12.$

(1) Dividing each side by 2, to make x^2 have 1 for its coefficient,

$$x^2 - \frac{5}{2}x = 6.$$

(2) Adding $(\frac{5}{4})^2$ to each side,

$$x^2 - \frac{5x}{2} + \frac{25}{16} = \frac{25}{16} + 6, \text{ or } \frac{121}{16}. \quad \textbf{Why?}$$

(3) Extracting the square root of each side,

$$x - \tfrac{5}{4} = \tfrac{11}{4}, \text{ or } -\tfrac{11}{4}.$$

(4) Using $+\frac{11}{4}$, $x = \frac{16}{4}$, or 4.

(5) Using $-\frac{11}{4}$, $x = \frac{5}{4} - \frac{11}{4}$, or $-\frac{3}{2}$.

(6) Check: The pupil should check the example.

EXERCISE 149

Solve and check each of these equations:

1. $2x^2 + 5x = 7$
2. $3x^2 + 4x = 16$
3. $2y^2 + y = 19$
4. $2x^2 + x = 10$
5. $2y^2 - y = 6$
6. $2y^2 - 10y = -8$
7. $3x^2 + 5x = 8$
8. $5x^2 + x = \frac{7}{4}$
9. $3y^2 - y = \frac{2}{3}$
10. $2x^2 - x - 1 = 0$
11. $3b^2 - 2b = 8$
12. $4a^2 + a = 5$
13. $5y^2 - 10y = 15$
14. $c^2 - \frac{1}{2}c = 3$
15. $y^2 - \frac{1}{3}y = 8$
16. $2p^2 - \frac{1}{3}p = 35$

EXERCISE 150

Solve and check.

1. $2x^2 + 10x = 72$
2. $3a^2 + 6a = 45$
3. $\frac{x^2}{2} + \frac{x}{4} = 9$

   HINT: Get rid of fractions.
4. $y^2 - 40 = 8y$
5. $\frac{2x^2}{3} - x = 3$
6. $\frac{x}{2} - \frac{3}{x} = \frac{47}{10}$
7. $\frac{x^2}{2} + \frac{3x}{5} = \frac{31}{2}$
8. $3y^2 + 5y = 22$
9. $5b^2 + 16b + 3 = 0$
10. The difference of two numbers is 4, and the sum of their squares is 210. Find the numbers.
11. A farmer has a square wheatfield containing 10 acres. In harvesting the wheat, he cuts a strip of uniform width around the field. How wide a strip must be cut in order to have the wheat half cut?
12. Divide 20 into two parts whose product is 96.
13. The sum of two numbers is 20, and the sum of their squares is 208. Find the numbers.
14. I went to the grocery for oranges. The clerk said they had advanced 10 cents per dozen. I got $\frac{1}{2}$ dozen fewer oranges for a dollar. What was the original price per dozen?
15. A piece of tin in the form of a square is taken to make an open-top box. The box is made by cutting out a 3-inch square from each corner of the piece of tin and folding up the sides. Find the length of the side of the original piece of tin if the box contains 243 cubic inches.

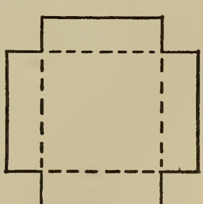


FIG. 143

16. A rectangular park 56 rods long and 16 rods wide is surrounded by a boulevard of uniform width. Find the width of this street if it contains 4 acres.

17. The members of a high-school class agreed to pay $8 for a sleigh ride. As 4 were obliged to be absent, the cost for each of the rest was 10 cents greater than it otherwise would have been. How many intended to go on the sleigh ride?

III. HOW TO SOLVE QUADRATIC EQUATIONS GRAPHICALLY

Section 171. Graphs show the relation between variables. We have frequently pointed out that graphs are used to represent relationship between related variables. Then, to solve a quadratic equation graphically, we must recognize these **variables** and **plot specific values for them.** Consider the equation:

$$x^2 - 8x + 12 = 0.$$

The expression $x^2 - 8x + 12$ may have many values, depending upon the value of x. Thus we have two related variables: **x itself is one variable, and $x^2 - 8x + 12$ is the other one.** To represent graphically the many values of $x^2 - 8x + 12$, we must assign values to x, and compute the corresponding values of $x^2 - 8x + 12$, *i.e.* we must tabulate the related values of the two variables.

Table 27 gives these values.

TABLE 27

If x is	0	1	2	3	4	5	6	7	-1	-2
then $x^2-8x+12$ is	12	5	0	-3	-4	-3	0	5	21	32

Next, we plot the various values of these two variables. For convenience, we plot the unknown x, on the horizontal axis, and the expression $x^2 - 8x + 12$ on the vertical axis. Locating the points represented by the pairs of values, (0, 12), (1, 5), (2, 0), etc., from the table, we obtain as the graphic representation of $x^2 - 8x + 12$, the curve of Fig. 144. From this graph, what is the value of $x^2 - 8x + 12$ when $x = 2$? when $x = 4$? when $x = 6$? when $x = 7$? Does the graph show for what values of x the expression equals 0? If it does, then it solves the equation $x^2 - 8x + 12 = 0$.

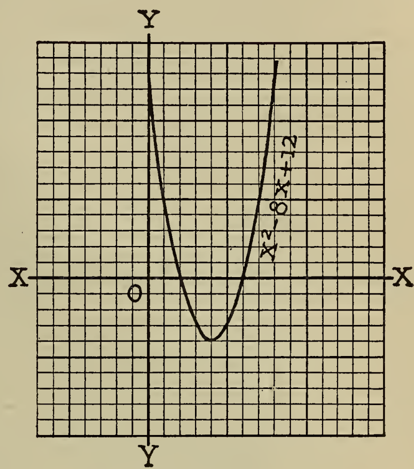


FIG. 144

This graph should emphasize the fact that in the quadratic equation $x^2 - 8x + 12 = 0$, the left side, $x^2 - 8x + 12$, may have many values, depending upon the value of x, and that this expression is 0 for two particular values of x; namely, when $x = 2$ and when $x = 6$. **These are the points at which the curve cuts the *x*-axis.**

A study of the graphical solution of another quadratic equation, $x^2 - 2x - 35 = 0$, Fig. 145, will make this method of solving quadratics still clearer.

By referring to it, you will be able to answer the following questions.

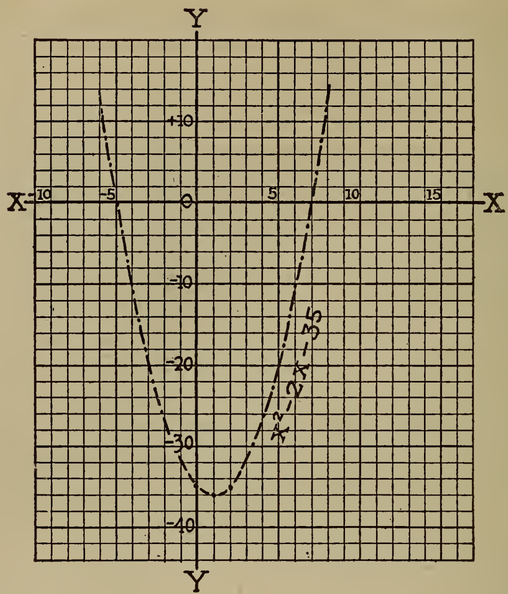


FIG. 145

1. How many different values may the expression $x^2 - 2x - 35$ have?
2. When is the value of the expression zero? *i.e.* where does the curve cross the x-axis?
3. Could the curve be drawn accurately by tabulating only two or three pair of values? Why?

Summary of steps in the graphical solution of quadratic equations. A study of these illustrations suggests the following:

1. **Graph the quadratic expression which forms one member of the equation. (The other member should be zero.)**

2. **From the graph find for what values of x the expression is zero. In other words, *find at what value of x the graph cuts the x-axis.* These values are the values of x which will *satisfy* the equation.**
3. **Check your result by substituting these values of the unknown in the original equation to see if they do *satisfy* it.**

EXERCISE 151

PRACTICE IN SOLVING QUADRATIC EQUATIONS GRAPHICALLY

Solve the following quadratic equations by drawing graphs of the expressions:

1. $x^2 - 5x - 14 = 0$
2. $x^2 + 3x = 40$
3. $y^2 - y = 20$
4. $2x = 48 - x^2$
5. $x^2 - 6x + 9 = 0$
6. $2y^2 + 5y + 3 = 0$
7. $x^2 + 8x + 16 = 0$
8. $y^2 + 3y + 1 = 0$
9. $(x - 2)^2 + 6x = 12$
10. $x^2 + 4 = 0$

In these examples, did you find any graph that did *not* cut the x-axis in two places? What conclusion would you draw if the graph **just touched** the x-axis? if it did not even touch it?

What seem to be the disadvantages of the graphical method of solving quadratic equations? the advantages?

REVIEW EXERCISE 152

1. By substituting any value for x in $x^2 - 1$, $2x$, and $x^2 + 1$, show that the three numbers which result are sides of a right triangle.

2. If the sides of a right triangle are 6 inches and 8 inches, then the hypotenuse must be __?__ inches.

3. How can you tell when a triangle is a right triangle without measuring its angle? Is the triangle whose sides are 5, 12, and 13 a right triangle? Why?

4. What is the area of an equilateral triangle each of whose sides is 30 inches?

5. How would you find the side of a square which had the same area as a circle with a radius of 12 inches?

6. Write a formula for b if a, b, and c are the altitude, base, and hypotenuse of a right triangle; similarly, a formula for a.

7. Express the hypotenuse of a right triangle whose altitude exceeds its base by 6 inches.

8. When is it impossible to find the square root of a number?

9. The sum of two numbers is $\frac{19}{6}$ and their product is $\frac{5}{3}$. Find the numbers.

10. The hypotenuse of a right triangle is 25 feet. Find the other two sides if you know that their sum is 35 feet.

11. A piece of wire 30 inches long is bent into the form of a right triangle whose hypotenuse is 13 inches. Find the other sides of the triangle.

12. The area and the perimeter of a rectangle are each 25. What are its dimensions?

13. A photograph, 8 inches by 10 inches, is enlarged until it covers twice the original area, keeping the ratio of the length to the width unchanged. Find the sides of the enlarged photograph.

14. In placing telephone poles between two places, it was found that if the poles were set 10 feet farther apart than originally planned, 4 poles fewer per mile were needed. How far apart were the poles placed at first?

15. Solve the equation

$$(2x+1)(x-2)=(x+2)(x-1)+5.$$

16. The altitude of a right triangle exceeds the base by 7. The hypotenuse is 13. Find the base and altitude.

17. Using the formula $s=16t^2$ for the distance, s, an object will fall in t seconds; determine how long it would take a stone to reach the ground if dropped from a height of one mile. (s is measured in feet.)

18. Solve for x: $x+1=20x^2$.

19. The area of the rectangular flag in a certain school is 98 sq. ft. and its perimeter is 42 ft. Solve for its dimensions.

20. How much area will a piece of wire 100 ft. long inclose if bent in the shape of a square? if bent in a circle?

21. Express the area formed by bending a piece of wire 10 ft. long into a square; by bending it into a circle.

22. Solve for v and R: $F=\frac{mv^2}{R}$.

23. Solve for M and v: $E = \frac{Mv^2}{2}$.

24. From the formula $2fs = v^2 - u^2$, derive a formula for v.

25. $S = Vt - \frac{1}{2}gt^2$ is a formula used by artillery officers. Derive a formula for V.

26. The distance, d (in miles), which one can see out on the ocean from the height, h (in feet), is expressed by the formula $d = 1.22\sqrt{h}$. How far could you see from a height of 50 feet? How high would one have to be to see a distance of 8 miles?

27. Two travelers leave a cross roads at the same time, one going due east 6 miles per hour, and the other due south 8 miles per hour. In how many hours will they be 40 miles apart?

28. A tourist has a trunk 32 inches long. It will just permit an umbrella 38 inches long to lie diagonally on the bottom. A trunk how much longer would be needed if he wanted to carry a gun 6 inches longer than the umbrella?

29. How many rods of fence will be required to inclose a square field containing 40 acres?

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